Partial Fraction Decomposition by Division

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In this note, we present a method for the partial fraction decomposition of two algebraic functions: (i) \( f(x)/(ax + b)^t \) and (ii) \( f(x)/(px^2 + qx + r)^t \), where \( f(x) \) is a polynomial of degree \( n \), \( t \) is a positive integer, and \( px^2 + qx + r \) is an irreducible \((q^2 < 4pr)\) quadratic polynomial. Our algorithm is relatively simple in comparison with those given elsewhere [1, 2, 3, 4, 5, 6, 7, 8]. The essence of the method is to use repeated division to re-express the numerator polynomial in powers of the normalized denominator. Then upon further divisions, we obtain a sum of partial fractions in the form \( A_i/(ax + b)^t \) or \( (B_j x + C_j)/(px^2 + qx + r)^t \) for the original function.

For (i), we let \( c = b/a \), and express \( f(x) \) as follows:

\[
f(x) = A_n(x + c)^n + A_{n-1}(x + c)^{n-1} + \cdots + A_2(x + c)^2 + A_1(x + c) + A_0,
\]

where each \( A_i \) is a real coefficient to be determined. Then the remainder after we divide \( f(x) \) by \( x + c \) gives the value of \( A_0 \). The quotient is

\[
q_1(x) = (x + c)[A_n(x + c)^{n-2} + A_{n-1}(x + c)^{n-3} + \cdots + A_3(x + c) + A_2] + A_1.
\]

If we now divide \( q_1(x) \) by \( x + c \), we see that the next remainder is \( A_1 \) and the quotient is

\[
q_2(x) = (x + c)[A_n(x + c)^{n-3} + A_{n-1}(x + c)^{n-4} + \cdots + A_3] + A_2.
\]

Continuing to divide in this manner \( n - 1 \) times, we get the quotient \( q_{n-1}(x) = A_n(x + c) + A_{n-1} \). Finally, dividing \( q_{n-1}(x) \) by \( x + c \), we obtain the last two coefficients, \( A_{n-1} \) and \( A_n \). Thus, it follows that

\[
\frac{f(x)}{(ax + b)^t} = \frac{1}{a^t} \left[ \frac{A_n}{(x + c)^{t-n}} + \frac{A_{n-1}}{(x + c)^{t-n+1}} + \cdots + \frac{A_1}{(x + c)^{t-1}} + \frac{A_0}{(x + c)^t} \right]. \tag{1}
\]

For example, to find the partial fraction decomposition of \((x^4 + 2x^3 - x^2 + 5)/(2x - 1)^3\), we use \( c = -1/2 \) and perform synthetic division to obtain \( A_0 \) through \( A_n \).

\[
\begin{array}{cccccc}
1/2 & 1 & 2 & -1 & 0 & 5 \\
1/2 & 5/4 & 1/8 & 1/16
\end{array}
\]

\[
\begin{array}{cccc}
1 & 5/2 & 1/4 & 1/8 \\
1/2 & 3/2 & 7/8 & \Rightarrow A_1
\end{array}
\]

\[
\begin{array}{cccc}
1 & 3 & 7/4 & 1 \\
1/2 & 7/4 & \Rightarrow A_1
\end{array}
\]

\[
\begin{array}{cccc}
1 & 7/2 & 7/2 & \Rightarrow A_2 \\
1/2
\end{array}
\]

\[
A_4 \Rightarrow 1 & 4 & \Rightarrow A_3
\]
Substituting the coefficients into (1), we have

\[
\frac{x^4 + 2x^3 - x^2 + 5}{(2x - 1)^5} = \frac{1}{2^5} \left[ \frac{1}{(x - 1/2)} + \frac{4}{(x - 1/2)^2} + \frac{7/2}{(x - 1/2)^3} + \frac{1}{(x - 1/2)^4} + \frac{81/16}{(x - 1/2)^5} \right]
\]

\[
= \frac{1}{16} \cdot \frac{1}{x - 1} + \frac{1/2}{(2x - 1)^2} + \frac{7/8}{(2x - 1)^3} + \frac{1/2}{(2x - 1)^4} + \frac{81/16}{(2x - 1)^5}.
\]

For (ii), we let \( u = q/p, \ v = r/p \), and express \( f(x) \) in the following form:

\[
f(x) = B_{(n-1)/2}(x^2 + ux + v)^{(n-1)/2} + B_{(n-3)/2}(x^2 + ux + v)^{(n-3)/2} + \ldots + B_1(x^2 + ux + v) + B_0,
\]

where each coefficient \( B_k, \ k = 0, 1, \ldots, (n - 1)/2 \), is a linear function of \( x \), and where we assume that \( n \leq 2t - 1 \). In this case, dividing \( f(x) \) and each successive quotient by \( x^2 + ux + v \) as described above, we obtain

\[
\frac{f(x)}{(px^2 + qx + r)^t} = \frac{1}{p^t} \left[ \frac{B_{(n-1)/2}}{(x^2 + ux + v)^{t-(n-1)/2}} + \frac{B_{(n-3)/2}}{(x^2 + ux + v)^{t-(n-3)/2}} + \ldots + \frac{B_0}{(x^2 + ux + v)^t} \right].
\]  

For instance, take the rational function \((x^5 - 4x^4 + 3x^2 - 2)/(x^2 - x + 2)^3\). Then \( u = -1 \) and \( v = 2 \). Since \((n - 1)/2 = 2\), we let

\[
x^5 - 4x^4 + 3x^2 - 2 = (Mx + N)(x^2 - x + 2)^2 + (Kx + L)(x^2 - x + 2) + Ix + J.
\]

Since most students are not familiar with the synthetic division technique when the divisor is a quadratic polynomial, long division can be used in place of the following computation to find the coefficients \( I, J, K, L, M, \) and \( N \).

\[
\begin{array}{cccc|ccc}
1 & -4 & 0 & 3 & 0 & -2 & 1 & -3 & -5 & 4 \\
1 & -1 & 2 & | & 1 & -1 & 2 \\
-3 & -2 & 3 & | & -5 & 9 & 0 \\
-3 & 3 & 6 & | & -5 & 5 & -10 \\
\hline
-5 & 9 & 0 & | & 4 & 10 & -2 \\
-5 & 9 & 0 & | & 4 & -4 & 8 \\
\hline
14 & -10 & 4 & | & 14 & -10 & 4
\end{array}
\]
Substituting the coefficients in (2) (note that \( t = (n - 1)/2 = 1 \)) gives

\[
\frac{x^5 - 4x^4 + 3x^2 - 2}{(x^2 - x + 2)^3} = \frac{x - 2}{x^2 - x + 2} + \frac{-9x + 8}{(x^2 - x + 2)^3} + \frac{14x - 10}{(x^2 - x + 2)^3}.
\]

Note also that \( x^2 - x + 2 = (x - 1/2)^2 + 4/7 \). On the right hand side of the above expression, replacing the coefficients \(-2, 8, \text{ and } -10\) in the numerators by \(-2 + 1/2M, 8 + 1/2M, \text{ and } -10 + 1/2M\), respectively, we get

\[
\frac{x^5 - 4x^4 + 3x^2 - 2}{((x - 1/2)^2 + 7/4)^3} = \frac{(x - 1/2) - 3/2}{(x - 1/2)^2 + 7/4} + \frac{-9(x - 1/2) + 7/4}{((x - 1/2)^2 + 7/4)^2} + \frac{14(x - 1/2)^2 - 3}{((x - 1/2)^2 + 7/4)^3}.
\]

This last expression is an easily antidifferentiable form.

References


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**An Elegant Mode for Determining the Mode**

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For any probability distribution, the mode, like the mean and median, is a measure of central tendency. Geometrically, it represents the relative maximum of the probability density function (pdf) and thus is the most striking feature in the curve’s topogra-