A Nonstandard Approach to Cramer's Rule

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Most textbooks in linear algebra develop Cramer's rule via the adjoint matrix. Therefore, the following approach may be worth noting.

Cramer's rule. If the coefficient matrix A of the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

has nonzero determinant, then the system has a unique solution

$$x_j = \frac{d_j}{d} \qquad (j = 1, 2, \ldots, n), \tag{2}$$

where d is the determinant of A, and d_j is the determinant of the $n \times n$ matrix obtained from A by replacing the jth column of A with the column of constants.

We first show that if $d \neq 0$, the system has a solution. Begin by considering the $(n+1) \times (n+1)$ determinant

$$\begin{vmatrix} b_j & a_{j1} & a_{j2} & \cdots & a_{jn} \\ b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_j & a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Since the first row and the $(j+1)$th row are equal, the determinant is zero. Therefore, expansion of the determinant along the first row gives

$$b_j d - a_{j1}d_1 - a_{j2}d_2 - \cdots - a_{jn}d_n = 0,$$

or

$$a_{j1}\left(\frac{d_1}{d}\right) + a_{j2}\left(\frac{d_2}{d}\right) + \cdots + a_{jn}\left(\frac{d_n}{d}\right) = b_j. \tag{3}$$

Since (3) holds for each j $(j = 1, 2, \ldots, n)$, a solution of (1) in the form of (2) does exist.

Now suppose that $(x_1, x_2, \ldots, x_n)$ is any solution of (1). Then

$$x_j d = \begin{vmatrix} x_j & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ 0 & a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}. \tag{4}$$

In (4), replace column 1 by column $1+\sum_{i=1}^{n}\{x_i$ times column$(i+1)\}$ and expand along the first row. This yields

$$x_j d=\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ b_1 & a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$=(-1)^2\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$=d_j.$$

Since $d\neq 0$, we obtain $x_j=\dfrac{d_j}{d}$ $(j=1,2,\ldots,n)$.

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Pascal Triangles and Combinations Where Repetitions Are Allowed

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For some simple, visual representations of any entry in Pascal's triangle, superimpose an inverted Pascal triangle with the apex at the desired entry. Then the value of the entry can be found by summing the product of the entries of any row of the original triangle by the corresponding overlapping entries in the inverted triangle. This is illustrated below, where the circled numbers are the entries of the inverted Pascal triangle.

$$\begin{aligned} 10(1) &= 4(1)+6(1) \\ &= 1(1)+3(2)+3(1) \end{aligned}$$

$$\begin{aligned} 126(1) &= 70(1)+56(1) \\ &= 35(1)+35(2)+21(1) \\ &= 15(1)+20(3)+15(3)+6(1) \\ &= 5(1)+10(4)+10(6)+5(4)+1(1) \end{aligned}$$