Bhatia [**2**] describes two applications of Look Up and Scream. The first involves directional antennas in a wireless network. Using a certain protocol, the antennas correspond to players in Look Up and Scream, with each player who screams corresponding to an antenna that can communicate with another antenna. The second involves peers in a specially designed peer-to-peer system in which resources are shared between two peers only if each peer has a resource that the other peer wants. A pair of players who scream corresponds to an exchange of resources between two peers.

Acknowledgment The author thanks the First Presbyterian Church of Ashland for supporting the mission trip to Traverse City that introduced him to Look Up and Scream and his wife for planning the mission trip.

REFERENCES

1. B. J. Balas and C. W. Connor, Look Up and Scream: Analytical difficulties in improv comedy, *Journal of Recreational Mathematics* **33**(1) (2004–2005) 32–38.
2. Swapnil P. Bhatia, Look Up and Scream: Analytical difficulties resolved! *Journal of Recreational Mathematics* **34**(1) (2005–2006) 12–23.
3. Saeed Ghahramani, *Fundamentals of Probability with Stochastic Processes*, 3rd ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2005.
4. Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics: An Applied Introduction*, 5th ed., Pearson, Boston, 2004.
5. Fred S. Roberts, *Applied Combinatorics*, Prentice Hall, Engelwood Cliffs, NJ, 1984.
6. Neil Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.

Summary In the game Look Up and Scream, players stand in a circle, close their eyes, and on the count of three, open their eyes, with each player looking directly at another player. If two players look directly at each other, they scream and are out of the game. In this paper, the author derives a formula for the probability that there are y pairs of yells when n people play a round of the game. Using this formula, the author derives formulas for the mean and variance of the number of pairs of yells and demonstrates how to calculate the mean rounds a game will last when starting with n players. The author also presents alternative derivations for the mean and variance of the number of pairs of yells.

Solving the Noneuclidean Uniform Circular Motion Problem by Newton's Impact Method

ROBERT L. LAMPHERE
Elizabethtown Community and Technical College
Elizabethtown, KY 42701
Robert.Lamphere@kctcs.edu

Sir Isaac Newton used a polygonal approximation method to show that the magnitude of the centripetal force that a particle experiences when uniformly revolving around a circle is

$$-\frac{mv^2}{r}, \tag{1}$$

where m is the particle's mass, v its uniform velocity, and r the circle's radius. In this note we use the same polygonal approximation method to prove that in noneuclidean

Math. Mag. **83** (2010) 366–369. doi:10.4169/002557010X529789.

(hyperbolic) geometry the magnitude of the centripetal force for a particle revolving uniformly around a circle of radius r on a hyperbolic plane of curvature -1 is

$$-\frac{mv^2}{\tanh r}. \tag{2}$$

In an earlier work [**3**], we used Newton's dynamic argument to prove (2). In that proof we used Galileo's basic law of falling bodies. In this paper we prove (2) simply by paralleling Newton's proof [**2**, p. 47] of (1), in current mathematical notation. Newton's proof consists of replacing the circular path by a n-sided regular polygon as shown in FIGURE 1(a). As the particle travels along this polygonal path, it repeatedly collides elastically with the circle, resulting in a impulsive force acting on the particle, a force directed towards the center of the circle. Newton computes the sum of these forces and then he lets the number of sides of the polygon increase to get (1). It is to be noted that all the geometric theorems, except for the *Law of Cosines*, that are employed in deriving (1) come from *absolute geometry*, the set of theorems that follow from Euclid's postulates other than the parallel postulate.

We want to show that the noneuclidean uniform circular motion problem can be solved with the same methods that Newton used to solve the same problem in Euclidean space.

In our proof of (2), we assume, since we cannot conduct noneuclidean experiments in Euclidean space, that Newtonian mechanics holds in any infinitesimal region of an noneuclidean plane. This is a reasonable assumption to make because an infinitesimal neighborhood of any point on a noneuclidean plane is euclidean [**1**, pp. 111 and 152]. It follows from this assumption, that we can prove (2) using the same physical arguments that Newton used to prove (1).

The geometry Before proving (2), we give some geometric results needed in our proof. Noneuclidean and Euclidean geometries share many theorems and constructions —everything except those that depend on the parallel postulate. The common ground among the geometries is called *absolute geometry*. For example, the theorem that a tangent to a circle is perpendicular to the circle's radius at the point of contact is a theorem of absolute geometry. So too is the side-angle-side congruency theorem. An example of a common construction is inscribing various regular polygons within a circle. These two theorems and this construction from absolute geometry, which are used to prove (1), can be used to prove (2).

The Law of Cosines can be used to prove (1), but that law depends on the parallel postulate. In hyperbolic geometry the Law of Cosines takes the form [**1**, pp. 102–104]

$$\cosh(BC) = \cosh(AB)\cosh(AC) - \sinh(AB)\sinh(AC)\cos\alpha,$$

where BC, AB, and AC are the lengths of the sides of the triangle ABC and $\alpha = \angle BAC$, which is different from the Euclidean form.

In our proof of (2), we use this hyperbolic law, which is the only difference between our proof and Newton's.

Proof by impact. To begin the proof, we first replace the circular path by an n-sided regular polygon path as shown in FIGURE 1(a), thus replacing the continuous motion by one with n collisions. This polygonal path is inscribed in the fixed circle of radius r centered at S. Let BC and CD be any two adjacent sides of the inscribed polygon. Draw EF tangent to the circle at C. Since triangles SCD and SCB are congruent and $EF \perp SC$, $\angle BCF = \angle DCE$.

Why would a sequence of elastic collisions produce such a path? We base our reasoning on the assumption that Newtonian mechanics holds within the infinitesimal

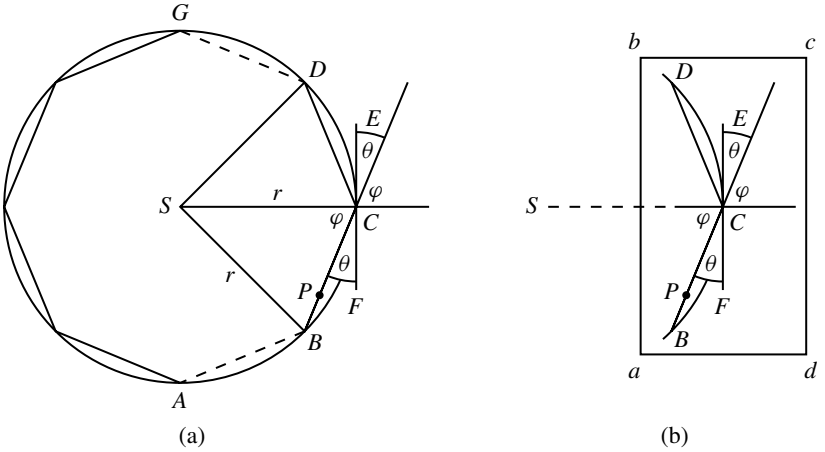

Figure 1 Particle P collides with the hyperbolic circle at C

region where the collision occurs. If the particle collides elastically with the circle at C, it is reflected off the tangent EF at C in such a way that $\angle BCF$ (incident angle) is equal to the angle of reflection and it has a rebound speed of v. But $\angle BCF = \angle DCE$, thus making $\angle DCE$ the reflection angle. Therefore, the particle will be reflected along CD with speed v. The same argument equally applies to the other collision points. This shows that the particle will continue to travel along the regular polygon's perimeter of FIGURE 1(a) with constant speed v after each collision.

Next we compute the force that the particle experiences when colliding with the circle at C. Again recall that Newtonian mechanics is assumed to hold in any infinitesimal region about point C (FIGURE 1(b)). Before colliding with the circle, the particle's linear momentum, relative to the rectangular axes CS and CE, is $\vec{M}_{BC} = (mv\sin\theta, mv\cos\theta)$, and after the collision it becomes $\vec{M}_{CD} = (-mv\sin\theta, mv\cos\theta)$ where $\theta = \angle BCF$. Therefore, the particle's total change of momentum, denoted by $\Delta\vec{\wp}$, is given by $\Delta\vec{\wp} = \vec{M}_{CD} - \vec{M}_{BC} = (-2mv\sin\theta, 0)$. Note that $\Delta\vec{\wp}$ points inward along the radius. Now let $\vec{F}$ be the force that the particle experiences while colliding with the circle, a force that acts for a very short time duration Δt. By Newton's second law of motion, we have $\Delta\vec{\wp} = \vec{F}\Delta t$, and so $\vec{F}\Delta t = (-2mv\sin\theta, 0)$. The name *impulse* is given to the product $\vec{F}\Delta t$ and its magnitude is given by

$$f\Delta t = -2mv\sin\theta, \tag{3}$$

where f denotes the magnitude of the force $\vec{F}$. It follows from the direction of $\Delta\vec{\wp}$ that the direction of the force f is toward the circle's center S along the radius SC.

Let $\Delta s = BC$ and $\varphi = \angle SCB$. Then applying the hyperbolic Law of Cosines to triangle SCB,

$$\cosh r = \cosh r\cosh(\Delta s) - \sinh r\sinh(\Delta s)\cos\varphi. \tag{4}$$

Since $SC \perp EF$, we have $\varphi = \frac{\pi}{2} - \theta$. Therefore, (4) can be rewritten, with the help of the half-angle formulas for hyperbolic trigonometric functions, as

$$\sin\theta = \frac{\sinh\frac{\Delta s}{2}}{\tanh r\cosh\frac{\Delta s}{2}}.$$

Next, substituting the last equation into (3) and rewriting, we find

$$f\Delta t = -m\frac{v\Delta s}{\tanh r}\left(\frac{\frac{\sinh\frac{\Delta s}{2}}{\frac{\Delta s}{2}}}{\cosh\frac{\Delta s}{2}}\right), \tag{5}$$

as the magnitude of the impulse at C, which is the same at the other collision points. Since there are n collision points, summing all the corresponding forces gives

$$fn\Delta t = -m\frac{v(n\Delta s)}{\tanh r}\left(\frac{\frac{\sinh\frac{\Delta s}{2}}{\frac{\Delta s}{2}}}{\cosh\frac{\Delta s}{2}}\right). \tag{6}$$

Next, let $n \to \infty$, so that the number of sides of the polygon increases without bound. Then $\Delta s \to 0$, $n\Delta s \to L$, and $n\Delta t \to T$, where L and T are the length of the circumference and the time the particle takes to travel around the circle, respectively. Thus, since $\lim_{x\to 0}(\sinh x)/x = 1$ (which easily follows from L'Hospital's rule), (6) becomes

$$fT = -m\frac{v}{\tanh r}L,$$

or, since $L = vT$,

$$f = -m\frac{v^2}{\tanh r},$$

the desired centripetal force. This completes the proof. ■

It is curious to note that in the hyperbolic plane, as the radius of a circular path becomes larger and larger, the limiting magnitude of force is mv^2, quite in contrast with the Newtonian case where the force dies out as the path of motion approaches a straight line. Perhaps this is not too surprising. If we fix a point on the circle and move the center farther and farther away along a line, the limiting shape is not a line, but a special curve called a *horocycle*; since this curve is not straight, force is required to keep the particle moving along it.

Elliptic geometry In the above proof, we can simply replace the hyperbolic Law of Cosines by the elliptic one to prove that the centripetal force in elliptic geometry (for a sphere of radius 1) is

$$-\frac{mv^2}{\tan r}.$$

In this case, when r approaches $\pi/2$, the force becomes 0, which is fitting since then the circular motion is along a great circle.

REFERENCES

1. H. S. Carslaw, *Non-Euclidean Plane Geometry and Trigonometry*, Longmans, Green, London, 1916. Reprinted by General Books LLC, 2010.
2. Florian Cajori, *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World*, Greenwood Press, New York, 1962.
3. Robert L. Lamphere, Solution of the direct problem of uniform circular motion in Non-Euclidean geometry, *Amer. Math. Monthly* **109** (2002) 650–655. doi:10.2307/3072429

Summary We compute the centripetal force exerted on a particle moving uniformly on the circumference of a noneuclidean circle using Newton's impact method.