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Wronskian Harmony

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Recently J. Wichmann in Osnabrück considered the Wronski-type determinants for the harmonic oscillators $\sin x$, $\sin 2x, \ldots, \sin kx$. He found these determinants for $k \leqq 7$ to be a coefficient times a power of $\sin x$, and hence suggested the formula

$$f_n(x) = \det\left(\frac{d^j(\sin(k+1)x)}{dx^j}\right)_{j,k=0,\ldots,n-1} = K_n(\sin x)^{S_n}, \tag{1}$$

with $S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and the coefficients K_n to be explored, e.g.

$$\begin{aligned}
f_1(x) &= \sin x\\
f_2(x) &= -2\sin^3 x\\
f_3(x) &= -16\sin^6 x\\
f_4(x) &= 768\sin^{10} x\\
f_5(x) &= 294912\sin^{15} x\\
f_6(x) &= -1132462080\sin^{21} x\\
f_7(x) &= -52183852646400\sin^{28} x\\
&\text{etc.}
\end{aligned}$$

The elusive formula cannot be completely trivial since it must depend on properties which the sine does not share with the cosine. The corresponding determinants with "cos" substituted for "sin" give several terms, e.g.,

$$\begin{vmatrix} \cos x & \cos 2x \\ -\sin x & -2\sin 2x \end{vmatrix} = 2\sin^3 x - 3\sin x$$

$$\begin{vmatrix} \cos x & \cos 2x & \cos 3x \\ -\sin x & -2\sin 2x & -3\sin 3x \\ -\cos x & -4\cos 2x & -9\cos 3x \end{vmatrix} = 16\cos x\sin^5 x - 40\cos x\sin^3 x.$$

On the other hand, if we substitute e^x for $\sin x$, then we obtain a formula similar to

(1), but this time the coefficients are easy to compute:

$$\det\left(\frac{d^j e^{(k+1)x}}{dx^j}\right)_{j,k=0,\ldots,n-1} = L_n (e^x)^{S_n}, \tag{2}$$

where S_n is the same as in (1) and $L_n = \prod_{j=0}^{n-1} j!$

A table of L_n for $n \leqslant 7$ is

TABLE 1.

n	1	2	3	4	5	6	7
L_n	1	1	2	12	288	34560	24883200

As the reader might have noticed, L_n is a divisor of K_n for $n \leqslant 7$. Furthermore, the quotient K_n/L_n is a power of -2, to be precise,

$$K_n = L_n \cdot (-2)^{S_{n-1}}. \tag{3}$$

The aim of this note is to prove the beautiful formula (1) with K_n defined by (3). We shall begin with the formula (2).

Proof of (2). In each column of the matrix the function is always the same, e.g., $e^{(k+1)x}$. Taking these factors outside we get the formula (2) with coefficient

$$L_n = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 4 & 9 & 16 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & 4^{n-1} & \cdots & n^{n-1} \end{vmatrix}. \tag{4}$$

This is the well-known Vandermonde determinant and is hence equal to:

$$\prod_{1 \leqslant j < k \leqslant n} (k-j) = \prod_{k=2}^{n} \prod_{j=1}^{k-1} (k-j) = \prod_{k=2}^{n} (k-1)! = \prod_{j=1}^{n-1} j! \tag{5}$$

However, it turns out that neither the formula (2) nor its proof shall be needed to show (1) and (3).

The proof of (1) and (3) depends on two lemmata to be proved later. The first lemma is due to Chebysheff:

LEMMA 1. *The trigonometric functions* $\sin nx$ *and* $\cos nx$ *can be expressed as polynomials in* $\sin x$ *and* $\cos x$ *as follows*:

$$\cos nx = 2^{n-1}(\cos x)^n + \cdots + (-1)^{n/2} \tag{6}$$

$$\sin nx = \sin x\left(2^{n-1}(\cos x)^{n-1} + \cdots\right) \tag{7}$$

with the term $(-1)^{n/2}$ *only to be included for n even.*

The polynomials are the Chebysheff polynomials.

LEMMA 2. *The Wronski determinants satisfy*

$$\det\left(\frac{d^j(fg_k)}{dx^j}\right)_{j,k=0,\ldots,n-1} = f^n \det\left(\frac{d^j g_k}{dx^j}\right)_{j,k=1,\ldots,n-1} \tag{8}$$

for $g_0 = 1$ *and* $f, g_1, \ldots, g_{n-1}$ *any functions.*

The difference between sine and cosine is disclosed by lemma 1 as the difference between the polynomials (6) and (7).

Proof of (1) and (3). In the matrix in (1) we write each function as a sum of polynomials as in (7).

$$f_n(x) = \begin{vmatrix} \sin x & \sin x(2\cos x) & \sin x(4\cos^2 x - 1) & \cdots & \sin x(2^{n-1}\cos^{n-1} x + \cdots) \\ \vdots & & \text{"derivatives"} & & \end{vmatrix}.$$

Starting in the third column, we notice that the term "$-\sin x$" gives a repetition of the first column. Hence it can be omitted. The following lower order terms must suffer the same fate as this one as we proceed from left to right. Hence, we are left with the highest powers of $\cos x$.

$$f_n(x) = \begin{vmatrix} \sin x & \sin x(2\cos x) & \sin x(4\cos^2 x) & \cdots & \sin x(2^{n-1}\cos^{n-1} x) \\ \vdots & & \dfrac{d^j(\sin x 2^k \cos^k x)}{dx^j} & & \\ & & & & j, k = 0, \ldots, n-1 \end{vmatrix}.$$

On this form lemma 2 applies. From (8) we get

$$f_n(x) = \sin^n x \cdot \det\left(\frac{d^j(2^k\cos^k x)}{dx^j}\right)_{j,k=1,\ldots,n-1} \tag{9}$$

The first row in the matrix of (9) is $(k = 1, \ldots, n-1)$:

$$\frac{d(2^k\cos^k x)}{dx} = 2^k \cdot k \cdot (\cos x)^{k-1} \cdot (-\sin x) = -2k \cdot \left(\sin x(2^{k-1}\cos^{k-1} x)\right). \tag{10}$$

This row is similar to the first row above except for the factor $-2k$. Taking this factor outside, we obtain the recursion formula:

$$f_n(x) = \prod_{k=1}^{n-1} (-2k)\sin^n x f_{n-1}(x) = (-2)^{n-1}(n-1)!\sin^n x f_{n-1}(x). \tag{11}$$

From this follows, with $f_0(x) = 1$,

$$f_n(x) = \prod_{k=1}^{n} (-2)^{k-1}(k-1)!\sin^k x = (-2)^{S_{n-1}} L_n (\sin x)^{S_n} = K_n(\sin x)^{S_n}. \tag{12}$$

This proof could not work for cosines in the obvious analogy. But including $1 = \cos 0x = (\cos x)^0$ it works. We have

$$g_n(x) = \det\left(\frac{d^j\cos(kx)}{dx^j}\right)_{j,k=0,\ldots,n} = 2^{S_{n-1}} L_{n+1}(-\sin x)^{S_n}. \tag{13}$$

Proof. Straightforward computation gives

$$g_n(x) = (-1)^n n! f_n(x).$$

Proof of lemma 1. The Chebysheff polynomials (6) are widely used, but the corresponding polynomials for $\sin nx$ (7) are less known. An easy computation gives both at once:

$$\cos nx + i\sin nx = (\cos x + i\sin x)^n = \sum_{j=0}^{n}\binom{n}{j}(\cos x)^{n-j}(i\sin x)^j$$

$$= \sum_{\substack{j=0\\ j \text{ even}}}^{n}(-1)^{j/2}\binom{n}{j}(\cos x)^{n-j}(1-\cos^2 x)^{j/2} +$$

$$i\sin x\sum_{\substack{j=0\\ j \text{ odd}}}^{n}(-1)^{(j-1)/2}\binom{n}{j}(\cos x)^{n-j}(1-\cos^2 x)^{(j-1)/2}$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}(\cos x)^{n-2k}\sum_{m=0}^{k}(-1)^m(\cos x)^{2m}\binom{k}{m} +$$

$$i\sin x\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}(\cos x)^{n-2k-1}\sum_{m=0}^{k}(-1)^m(\cos x)^{2m}\binom{k}{m}$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\cos^n x + \cdots + (-1)^{n/2} + i\sin x\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\cos^{n-1}x + \cdots$$

$$= 2^{n-1}\cos^n x + \cdots + (-1)^{n/2} + i\sin x\, 2^{n-1}\cos^{n-1}x + \cdots$$

where the term $(-1)^{n/2}$ only occurs for n even, and the dots stands for terms of lower order.

Proof of lemma 2. If $f(x) = 0$, then everything is zero. So we can assume $f(x) \neq 0$. Leibniz' formula for derivation is

$$\frac{d^j(fg)}{dx^j} = \sum_{k=0}^{j}\binom{j}{k}f^{(k)}g^{(j-k)}. \tag{14}$$

Using (14) row by row in the matrix we find that the last terms in (14) give us a part proportional to the first row.

$$\frac{f^{(k)}}{f}\cdot(f, fg_1, \ldots, fg_{n-1}) = \left(f^{(k)}, f^{(k)}g_1, \ldots, f^{(k)}g_{n-1}\right).$$

So these terms can be subtracted from each row except the first one. Then the determinant looks like this:

$$\begin{vmatrix} f & fg_1 & fg_2 & \cdots & fg_{n-1} \\ 0 & fg_1' & fg_2' & \cdots & fg_{n-1}' \\ \vdots & & \sum_{k=0}^{j-1}\binom{j}{k}f^{(k)}g_m^{(j-k)} & & \\ 0 & & & & \end{vmatrix}.$$

Now, the second to last term of the sum, i.e.,

$$\binom{j}{j-1} f^{(j-1)} g'_m$$

gives a part proportional to the second row as it looks now. Hence this part can be omitted from the third row and on. Continuing this way we end up with the determinant looking like

$$\begin{vmatrix} f & fg_1 & fg_2 & fg_3 & \cdots & fg_{n-1} \\ 0 & fg'_1 & fg'_2 & fg'_3 & \cdots & fg'_{n-1} \\ 0 & fg''_1 & fg''_2 & fg''_3 & & fg''_{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & fg_1^{(n-1)} & & & & fg_{n-1}^{(n-1)} \end{vmatrix}.$$

Taking the factor f outside and developing the determinant after the first column one gets lemma 2 immediately.

Residual Lifetimes in Random Parallel Systems

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You go to the Statue of Liberty and return with a dozen souvenir glasses. They begin to break soon thereafter, but the last glass in the set seems to linger on for years. Why is it that residual 'components' in such a random system last a surprisingly long time? In the autumn leaves drop from the tree in the front yard; most of the leaves are clustered close to the trunk, but a few come to rest much further distant than one might expect. Neutrons in a reactor are absorbed by atoms in the shielding material; bullets shot into a forest are stopped by trees. How are the residual neutrons/bullets —the ones that are stopped most distant from the source—distributed?

This note explores the distributions of such residual 'components'; in particular, why does the last component last so long (or outdistance all the others by so much)? Two models are considered in detail. The first consists of a system of components with exponentially distributed lifetimes; here the residual components are those with greatest lifetimes. Starting with a large number N at time 0 equations (11), (12), and (13) show how the expected number of surviving components C depends on certain 'benchmark' times; most notable is equation (11), which is an approximation to the number of surviving components at fraction α of the total time until the final component dies. Equation (21) is a recursive formula for the fraction of total system lifetime during which only one component is living. Equation (14) is analogous to (11), but applies to the two-dimensional 'leaf model' in which objects fall on a plane with positions independent and normally distributed.

Definition A *parallel system of N components* consists of N components which all begin operating at time 0. The components are labeled from 1 to N and the ith component has a lifetime T_i which is random. The random variables $T_1, \dots, T_N$ are independent, each with the same distribution function