where Σ_{M_2A} represents the sum being taken over M_2A and $\chi = \left|\frac{\xi-\eta}{2}\right|$, $\eta = \frac{c}{2^2} - \xi$. Repeating the process for each $n = 1,2,3,\dots,$ we conclude from (8) that

$$\begin{aligned} 0 &\le \left|\sum x^2 - 4\sum \alpha^2\right| = 2^n\left|\sum_{M_nA} \xi^2 - 4\sum_{M_nA}\chi^2\right| \\ &\le 2^n \sum_{M_nA} \xi^2 = 2^n \sum_{M_nA}\left(\frac{c}{2^n}\right)^2 \\ &\le \frac{\sum c^2}{2^n}. \end{aligned} \tag{9}$$

The claim in (3) is verified by making n large enough (i.e., $n \to \infty$).

References

[1] M. H. Bosman, S. J., Un chapitre de l'oeuvre de Cavalieri, Mathesis, 36 (1922) 336–373.
[2] C. Boyer, Cavalieri, Limit and Discarded Infinitesimals, Scripta Mathematica, 8 (1941) 79–91.
[3] C. H. Edwards Jr., The Historical Development of the Calculus, Springer-Verlag, New York, 1979.
[4] H. Eves, Slicing it Thin, The Mathematical Gardner, ed. David A. Klarner, Wadsworth International, Belmont, California, 1981, pp. 100–111, and Great Moments in Mathematics (before 1650), MAA Dolciani Mathematical Expositions no. 5, 1983, pp. 206–214.

Periodic Equilibria Under Periodic Harvesting

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A simple deterministic model for the growth of one population which is subject to periodic or seasonal harvesting due to fishing, hunting, or disease is given by the ordinary differential equation

$$\dot{x} = g(x) - h(t). \tag{1}$$

Here $x = x(t)$ is the size of the population at time t, $g(x)$ is a smooth function usually of the form $xf(x)$, where $f(x)$ is the per capita rate of growth, and $h(t)$ is a T-periodic function representing the harvesting effect. The case where $h(t)$ is a constant was discussed by the second author in an earlier note [**6**] for the case of a finite difference model, and by F. Brauer and the second author in [**1**] for the differential equation model.

An interesting question is the following: *what is the maximum number of T-periodic solutions equation (1) can have?* If $g(x)$ is a polynomial of degree 2 or 3 then the answer is 2 and 3, respectively. This is a not so well-known result which can be found in [**5**, pp. 102–119], and which is periodically rediscovered. For the more general case of a differential equation of the polynomial form

$$\dot{x} = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \; n \geqslant 4,$$

where $a_j(t)$, $j=0,\ldots,n-1$, are smooth T-periodic functions, the determination of an upper bound on the number of T-periodic solutions is an open question. For a recent discussion of this problem see the articles by A. L. Neto [**4**] and S. Shashahani [**8**].

Since the positive periodic solutions of (1) represent stable or unstable periodic equilibria sustained by the population under harvesting, an upper bound on the total number of periodic solutions would be useful. Fortunately, such a bound can often be obtained using the equation of variation to study the Poincaré map of (1).

Let $x(t,a)$ be the solution of (1) satisfying the initial conditions $x(0,a)=a$, and let I be the set of values of a for which the solution $x(t,a)$ exists on $0\leqslant t\leqslant T$. Then I is an open interval, and by uniqueness of solutions, $x(t,a)$ is strictly increasing on I as a function of a for fixed $t\in[0,T]$. Since

$$\frac{d}{dt}x(t,a)=g(x(t,a))-h(t),$$

it follows that if $\phi(t,a)=\dfrac{\partial}{\partial a}x(t,a)$ then $\phi(t,a)$ satisfies the variational equation

$$\frac{d}{dt}\phi(t,a)=g'(x(t,a))\phi(t,a),\quad \phi(0,a)=1,$$

where $g'=dg/dx$. Therefore $\phi(t,a)$ satisfies

$$\phi(t,a)=\exp\left\{\int_0^t g'(x(s,a))\,ds\right\}.$$

The Poincaré map associated with (1) is the map $a\to x(T,a)$ and we define

$$H(a)=x(T,a)-a,\quad a\in I.$$

Therefore, a zero of $H(a)$ corresponds to a periodic solution of (1) and conversely, if multiplicities are taken into account. From the previous analysis it follows that

$$H'(a)=\phi(T,a)-1=\exp\left\{\int_0^T g'(x(s,a))\,ds\right\}-1,$$

so if $n-1$ is an upper bound on the number of zeros of $H'(a)$ then (1) will have at most n periodic solutions. This leads to the following criterion:

If $g''(x)$ is either strictly negative or strictly positive for all x, then (1) will have at most two periodic solutions.

For, if the above holds, then $g'(x)$ is either strictly increasing or strictly decreasing, from which it follows that $H'(a)$ will be also since $x(s,a)$ is strictly increasing in a. Therefore $H(a)$ can have at most two zeros.

A straightforward stability analysis also shows that in the case where a_1 and a_2 are zeros of $H(a)$ with $a_1<a_2$ then the corresponding periodic solutions will be respectively, stable and unstable if $g'(x)>0$, whereas the stability will be reversed if $g'(x)<0$. In any case, once it is determined that periodic solutions may exist, a further analysis of the direction field of (1) or numerical analysis will help locate the periodic solutions, if any.

EXAMPLE 1. As a simple first example we consider the logistic equation with periodic harvesting

$$\dot{x}=rx\left(1-\frac{x}{K}\right)-(1+\varepsilon\cos t),\quad r,K>0,0<\varepsilon<1. \tag{2}$$

Here $g(x)=rx\left(1-\dfrac{x}{K}\right)$, and $g''(x)=-2r/K<0$, so there are at most two 2π-periodic solutions. Denoting the right side of equation (2) by $f(x,t)$ we see that $f(0,t)<0$, $f(K,t)<0$, and if $rK/4>1+\varepsilon$ then $f(K/2,t)>0$. Consequently, there is a (stable) periodic solution with initial value $K/2<a<K$, and an unstable one with initial value $0<a<K/2$.

A discussion of the above equation with no harvesting where $K = K(t)$ was also periodic (corresponding to a fluctuating environment) was given by B. D. Coleman, Y. Hsieh, and G. P. Knowles in [**2**]. See also [7] for a more simplified analysis including harvesting.

EXAMPLE 2. The following model for populations of the North American spruce budworm was given by D. Ludwig, D. D. Jones, and C. S. Holling [**3**]:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{\beta x^2}{\alpha^2 + x^2}, \quad r, K, \alpha, \beta > 0. \tag{3}$$

The second term on the right side of equation (3) models predation by birds and in the absence of predation, the growth is assumed to be logistic. If additional periodic harvesting (say, due to seasonal spraying) were to occur, then the equation would be of the form (1).

In this case

$$g''(x) = -2\left[\frac{r}{K} + \beta\alpha^2\frac{\alpha^2 - 3x^2}{(\alpha^2 + x^2)^3}\right].$$

For $x \geqslant 0$, $g''(x)$ will be negative if $r/K - \beta/4\alpha^2 > 0$, and there will be at most two periodic solutions. For appropriate values of the constants there will be a stable equilibrium point x_0 satisfying $K/2 < x_0 < K$ when there is no periodic harvesting—this can be easily seen by graphing the two expressions comprising the right hand side of the differential equation. Under small amplitude periodic harvesting, the equilibrium point will become a periodic solution.

References

[1] F. Brauer and D. A. Sânchez, Constant rate population harvesting: equilibrium and stability, Theoret. Population Biol., 8 (1975) 12–30.

[2] B. D. Coleman, Y. Hsieh, and G. P. Knowles, On the optimal choice of r for a population in a periodic environment, Math. Biosci., 46 (1979) 71–85.

[3] D. Ludwig, D. D. Jones, and C. S. Holling, Qualitative analysis of insect outbreak systems: the spruce budworm and forest, J. Animal Ecol., 47 (1978) 315–332.

[4] A. L. Neto, On the number of solutions of $dx/dt = \sum_{j=0}^{\infty} a_j(t)x^j, 0 \leqslant t \leqslant 1$, for which $x(0) = x(1)$, Invent. Math., 59 (1980) 69–76.

[5] V. A. Pliss, Nonlocal Problems of the Theory of Oscillations, Academic Press, New York, 1966.

[6] D. A. Sânchez, Populations and harvesting, SIAM Rev., 19 (1977) 551–553.

[7] ______, Periodic environments, harvesting and a Riccati equation, in "Nonlinear Phenomena in Mathematical Sciences," ed. V. Lakshmikantham, Academic Press, New York, 1982, pp. 883–886.

[8] S. Shashahani, Periodic solutions of polynomial first order differential equations, Nonlinear Anal. 5, (1981) 157–165.

The Maximum Brightness of Venus

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Even a casual observer may notice that the planet Venus, at times the dominant object in the evening sky, appears noticeably brighter at some times than at others. While reading a book on popular astronomy [**4**] one day, I came across the statement (with little explanation) that Venus is at its brightest when the illuminated portion of the apparent disk of the planet is 28% of the whole. Could that fact, I wondered, be determined theoretically? If you reflect for a moment, as I did, on the relative positions of the Earth, Venus and the Sun at various times, you should be able to see that, apart from the observer's local conditions, the brightness of Venus depends primarily