Proof Without Words: Viviani's Theorem with Vectors

The sum of the distances from a point P in an equilateral to the three sides of the triangle is independent of the position of P (and so equal to the altitude of the triangle.)

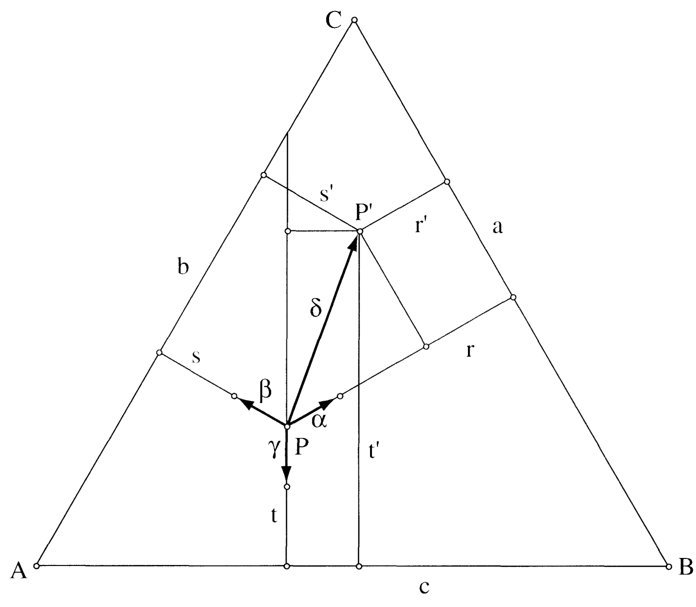
$$|\alpha| = |\beta| = |\gamma|$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha \cdot \delta + \beta \cdot \delta + \gamma \cdot \delta = 0$$

$$(r - r') + (s - s') + (t - t') = 0$$

$$r + s + t = r' + s' + t'$$



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Let π be 3

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And he (Hirum of Tyre) made a molten sea, ten cubits from the one brim to the other: it was round all about, and its height was five cubits: and a line of thirty cubits did compass it round about. 1 Kings 7:23

The literal interpretation of this Biblical passage is that Hirum constructed a hemispherical basin that had a diameter of 10 cubits (a cubit is approximately one half of a meter) and had a circumference three times that value. This ratio of the circumference of a circle to its diameter apparently contradicts results of the works of Archimedes who established that the ratio of the circumference of any circle to its diameter is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Since Archimedes provided a convincing argument for his values we are inclined to accept them as true and regard the numbers given to us in First Kings as approximate values whose error is due to rounding off. In this paper we do not write this off to a rounding error, but rather identify a setting where 3 is the correct value.

Archimedes' results were obtained in Euclidean geometry. Using alternate geometries we will establish that the ratio of the circumference of a circle to its diameter may take on a continuum of values, including three. In the next section we will discuss how Archimedes first determined his values. Then we will show how the ratio varies in spherical geometry. Finally, we discuss the possible values in hyperbolic geometry.

The results of Archimedes The computation of π has a long history [**2**]. Archimedes [**1**] first considered a regular polygon as inscribed within a circle and then as circumscribed about a circle, and thus was able to compute a lower approximation and an upper approximation for the ratio of the circumference of a circle to its diameter. He observed these ratios up to a polygon with 96 sides, and thus was able to conclude that the ratio of the circumference of any circle to its diameter is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Using Archimedes' technique and modern trigonometry we can compute even better approximations for this ratio.

Let us assume that a circle is divided into 360 degrees, and consider regular polygons having n sides (in our diagrams $n = 6$), where the length of each of the sides is 1 unit.

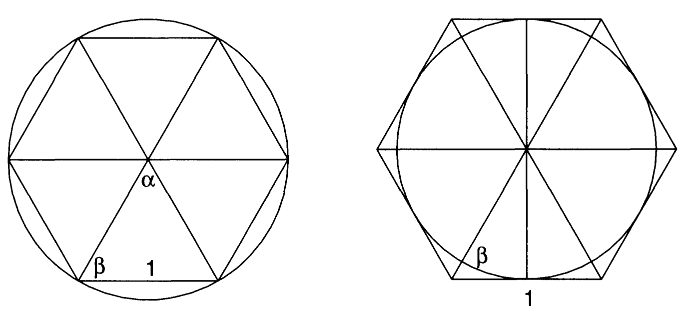

Figure 1 A circle circumscribed and inscribed by a regular polygon

From the easy observation that the measure of the angle α is $360°/n$, we find that the measure of the angle β is $90°(n-2)/n$. Now using the law of sines, we can see that the diameter of the circumscribed circle would be

$$2 \cdot \frac{\sin(90° \cdot (n-2)/n)}{\sin(360°/n)}.$$

The diameter of the inscribed circle is easily calculated to be

$$\tan(90° \cdot (n-2)/n).$$

We see that the ratio of the circumference of the circle to its diameter lies between ratio of the perimeter of the inscribed polygon to the circle's diameter and the ratio of the perimeter of the circumscribed polygon to the circle's diameter. For the remainder of the paper we will follow the universally accepted convention of using π as the ratio of the circumference of a circle to its diameter in Euclidean geometry. Computing the ratio of the perimeter of each polygon to the diameter of the circle, we derive the inequality

$$n \cdot \frac{\sin(360^\circ/n)}{2 \cdot \sin(90^\circ(n-2)/n)} < \pi < n \cdot \cot(90^\circ(n-2)/n).$$

We can now take the limits as n approaches infinity. The results yield what we already know, that each ratio approaches the same value, which is approximately 3.141592654.

The ratio π in spherical geometry From First Kings the ratio of the circumference to the diameter is easily computed to be 3. For the sake of academic curiosity let us assume that the ratio of the circumference to the diameter of the "molten sea" was indeed supposed to be 3. How can this be? An answer lies in the geometry under consideration. Archimedes proved his results in Euclidean geometry, the geometry of plane or flat surfaces. But suppose we were to use spherical geometry. Can we then produce circles in which the ratios of the circumferences to the diameters are indeed exactly 3?

In Euclidean geometry, the surface upon which measurements are made is the plane. In spherical geometry, the universal surface upon which measurements are made is the sphere. To perform calculations on the sphere we imagine the sphere embedded in three-dimensional Euclidean space. We may then use the results of Euclidean geometry to compute measurements on the sphere.

In the spherical setting, the ratio of circumference to diameter need not be π; in fact, it is easy to see how to produce a circle whose circumference-to-diameter ratio is 2. Simply choose the circle to be a great arc (equator), then the diameter of the circle (the shortest path between two antipodal points on the circle) is half the length of the great arc. So the ratio of the great arc to $1/2$ of a great arc is $1/(1/2) = 2$.

Since we now know that on a sphere the ratio of a circumference of a circle to its diameter need not be π, we turn our attention to all possible values for this ratio.

Let a circle with circumference C and diameter $2r$ lie on a sphere with radius R. Let α be the center of the circle, β the center of the sphere, γ a point on the circle, and δ the center of the circle as viewed in Euclidean geometry. Call ρ the Euclidean radius of the circle in the planar cross-section shown, and θ the angle $\angle\alpha\beta\gamma$, as in FIGURE 2.

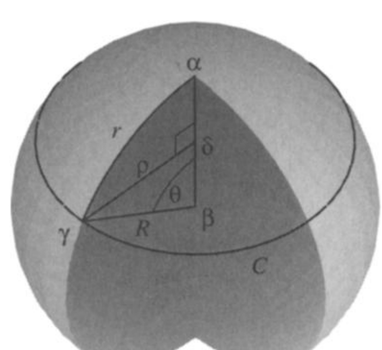

Figure 2 A circle on a sphere

We have $r = R\theta$, $\sin\theta = \rho/R$, and $2\rho\pi = C$, so that $\rho = R\sin\theta$ and $\theta = r/R$. Thus

$$C = 2\pi R \sin\frac{r}{R} = 2\pi\frac{r}{\theta}\sin\theta.$$

Let Π denote the ratio of the circumference of a circle to its diameter, which we represent as a function of θ. Since $2r$ is the diameter of the circle on the sphere, we can compute

$$\Pi(\theta) = \frac{C(\theta)}{2r} = \pi\frac{\sin\theta}{\theta}.$$

We now can compute $\Pi(\pi/2) = 2$, as we noted earlier. We can also compute $\Pi(\pi/6) = 3$, which is the ratio given in First Kings. We also note that the limiting value as the angle θ approaches 0 is

$$\lim_{\theta\to 0}\Pi(\theta) = \pi\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \pi.$$

We may allow θ to grow larger than $\pi/2$ and define the diameter of the circle to be twice the radius, which is the length of the arc from a point on the circle to the center, which we might as well call the North Pole. The graph of $\Pi(\theta) = \pi\sin(\theta)/\theta$ will represent all possible values for Π on the sphere.

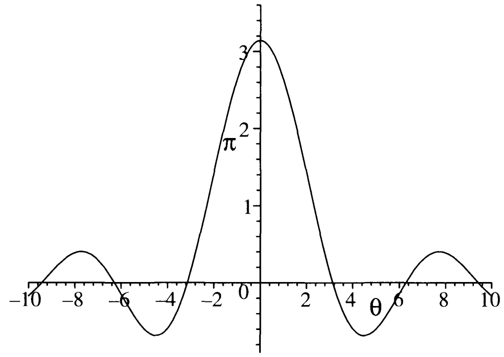

Figure 3 The graph of $\Pi(\theta) = \pi(\sin(\theta))/\theta$

As the terminus of angle θ passes the South Pole we will measure the circumference in the opposite direction, hence yielding the negative values for Π.

Also, we see that $\lim_{\theta\to\pm\infty}\pi(\sin\theta)/\theta = 0$. As θ grows larger the radius is wrapped once around the sphere for every multiple of 2π, but the absolute value of the circumference will never grow larger than the length of the equator.

The graph of our function suggests that a minimum value for Π occurs near ± 4.5. The standard calculus technique of setting the first derivative of $\Pi(\theta) = \pi(\sin\theta)/\theta$ equal to 0 yields the following results:

$$\Pi'(\theta) = \pi\frac{\theta\cos\theta - \sin\theta}{\theta^2} = 0 \Rightarrow \tan\theta = \theta,$$

so that $\theta \approx \pm 4.493409458$. We thus compute the minimum value to be approximately

$$\Pi(4.493409458) = \pi\frac{\sin 4.493409458}{4.493409458} = -.6824595706.$$

Thus we may conclude that for spherical geometry the ratio of the circumference of a circle to its diameter ranges over the values

$$-.6824595706 \leq \Pi < \pi \approx 3.141592654.$$

If we return to the values given to us in First Kings we may compute the size of the sphere on which the measurements of the molten sea are made. Using the equation $C = 2\pi R \sin r/R$, we solve for R when $C = 30$ and $r = 5$:

$$30 = 2\pi R \sin \frac{5}{R} \quad \text{or} \quad \frac{15}{\pi} = R \sin \frac{5}{R}.$$

We have no closed form solution for this equation, but the function

$$F(R) = R \sin\left(\frac{5}{R}\right) - \frac{15}{\pi}$$

is a continuously differentiable function for $R > 0$, so Newton's method would produce an accurate approximation. This takes only a few seconds using computer programs such as *Maple* or *Mathematica*, and we find that $R \approx 9.549296586$ cubits. Thus a circle at latitude 60° on a sphere of radius 9.549296586 cubits will have a circumference of 30 cubits and a diameter of 10 cubits.

The ratio π in hyperbolic geometry Spherical geometry is not the only alternative to the Euclidean plane. Any smooth two-dimensional surfaces in $\mathbb{R}^3$ might do just as well. Any such surface has an intrinsic measurement of curvature, which gives us an idea of how curved the surface is at each point. The curvature, or more precisely the Gaussian curvature, is computed as the product of two other quantities called the *principal curvatures* at a point. These principal curvatures are the maximum and minimum curvatures of the collection of one-dimensional arcs through that point. For a circle the curvature is $1/R$ where $|R|$ is the radius. We comment that R may be positive or negative depending as to whether we make the measurement from a vantage point inside the circle or outside the circle. Since the curvature of every arc on a sphere through any given point is $1/R$ where $|R|$ is the radius of the sphere, we have the curvature of the sphere to be the constant $K = 1/R^2$, which is always positive.

It is possible for surfaces to have negative curvature. The saddle point of a hyperbolic paraboloid is such an example. Since the surface is curving in a concave fashion in one direction and a convex fashion in the other direction, the maximal principal curvature will be positive and the minimal principal curvature will be negative. The Gaussian curvature at the saddle point is thus the product of a negative value and a positive value, which must be negative. O'Neill [**3**] is one standard reference.

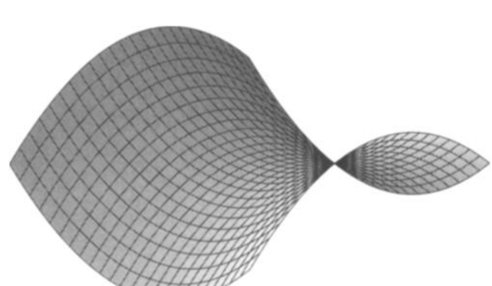

Figure 4 The hyperbolic paraboloid

We may now consider a space where at every point the principal curvatures are $1/R$ and $-1/R$, and hence the Gaussian curvature is the negative value $K = -1/R^2 = 1/(iR)^2$. We may consider this space to be a pseudo-sphere with an imaginary radius. The geometry on this space is known as hyperbolic geometry.

We can replace R with iR in our development of the function $\Pi(\theta)$ to get the corresponding formulas for hyperbolic geometry.

$$C = 2\pi i R \sin\frac{r}{iR} \quad \text{where } \frac{r}{R} = \theta.$$

Thus, we get a formula reminiscent of the spherical ratios,

$$C = 2\pi i r \frac{\sin\frac{\theta}{i}}{\theta} = 2\pi i r \frac{\sin(-i\theta)}{\theta}.$$

Apply the identity $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, to give us

$$C = 2\pi i r \frac{e^{-i^2\theta} - e^{i^2\theta}}{2i\theta} = 2\pi r \frac{e^{\theta} - e^{-\theta}}{2\theta} \equiv 2\pi r \frac{\sinh\theta}{\theta}.$$

Thus

$$\Pi(\theta) = \frac{C}{2r} = \pi\frac{\sinh\theta}{\theta}.$$

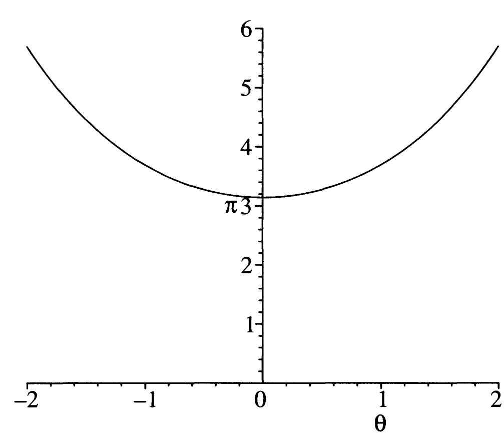

Figure 5 The graph of $\Pi(\theta) = \pi(\sinh\theta)/\theta$

The graph of $\Pi(\theta) = \pi(\sinh\theta)/\theta$ in FIGURE 5 reveals that Π takes on values greater than π in hyperbolic geometry. Easy limit computations produce

$$\lim_{\theta\to 0} \pi\frac{\sinh\theta}{\theta} = \pi \quad \text{and} \quad \lim_{\theta\to\pm\infty} \pi\frac{\sinh\theta}{\theta} = +\infty.$$

So we may conclude that, in hyperbolic geometry, Π takes on all values greater than π.

Conclusion The ancient Hebrews were certainly unaware of alternate geometries and were more concerned with the spiritual aspects of their lives than mathematical precision. Since it is highly unlikely that they would choose a sphere of approximately 9 meters in diameter on which to make their measurements, we can rationally conclude the discrepancies between First Kings and Archimedes is due to a very coarse approximation. But it is entertaining to realize that these measurements can be made exact by using the appropriate geometry.

Archimedes did not know the formal limit concept we use today, but he most surely knew the intuitive concept. Today the exact value of π is known to be the limit of the sequence produced by Archimedes. It is interesting to note that π is the limit of the ratio of circumferences of circles to their diameters in both the spherical and hyperbolic geometries. But this should not be surprising, since the limit is taken as the central angle approaches 0. If we imagine that the diameter of the circle is held constant, then the radius of the sphere or pseudo-sphere must approach infinity and the curvature approaches 0. Thus the Euclidean plane can be thought of as a sphere or pseudo-sphere with curvature 0.

Using our three geometries, π can be assigned any positive real value that you want, and even some negative values. We find it compelling to ponder the possibility of a geometry that would allow all negative values.

REFERENCES

1. Archimedes, Measurement of a circle, *The Works of Archimedes with Introductory Chapters by J. L. Heath*, Dover Publications, New York, 1912.
2. D. Castellanos, The ubiquitous π, this MAGAZINE, **61** (1988), 67–98 & 148–163.
3. Barrett O'Neill, *Elementary Differential Geometry*, Academic Press, San Diego, CA, 1966.

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circle. Thus, no points on the curved parts of the convex hull belong to any of the arcs whose lengths we are summing. Because the curved parts of the convex hull can be translated together to form a unit circle, their total length is 2π. Thus, our bound is improved to $2(n-1)\pi$. Combining this with our previous information, we have

$$\sum_{1 \le i < j \le n} \frac{8}{O_i O_j} < 2(n-1)\pi.$$

Dividing by 8 yields the desired inequality.