Linear Relations Between Powers of Terms in Arithmetic Progression

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Students aspiring to discover new and interesting results in mathematics need to learn that one of the keys to success is asking a good question. For example, when is it the case that the quadratic equation $ax^2 + bx - c = 0$ with a, b, and c positive integers that (in some order) form an arithmetic progression has rational solutions? Surprisingly, it turns out that

$$nx^2 + (n + r)x - (n + 2r) = 0$$

has rational solutions if and only if $n = r(F_{2m+1} - 1)$ and the solutions are $F_{2m}/(F_{2m+1} - 1)$ and $-F_{2m+2}/(F_{2m+1} - 1)$ independent of r where F_k denotes the kth Fibonacci number. This and similar results can be found in [**2**] and eventually lead to a remarkable result concerning Pell's equation in [**3**].

In this paper, we consider a somewhat related question that has an equally surprising answer and that provides an interesting and informative study that students can readily pursue. In particular, what can be said about the solutions to linear systems of equations whose coefficients are in arithmetic progression—systems like

$$\begin{aligned} 3x + 5y &= 7 \\ 6x + 9y &= 12 \end{aligned}$$

or

$$\begin{aligned} 2x + 5y + 8z &= 11 \\ x + y + z &= 1 \\ 3x + 2y + z &= 0\ ? \end{aligned}$$

It is immediate that the solution to the 2 by 2 system is $(-1, 2)$ and, surprisingly, this is so for any system of the form

$$\begin{aligned} ax + (a + d)y &= a + 2d \\ bx + (b + e)y &= b + 2e. \end{aligned}$$

That is to say, the equation

$$-a + 2(a + d) = a + 2d$$

is an identity.

Is there a similar result for 3 by 3 or larger systems of the same type? Unfortunately, the answer is no; any such system is necessarily dependent. Still one might ask, "What if, for three independent linear equations, the coefficients are *squares* of numbers in arithmetic progression?" When it turns out that the solution for such a system is $(1, -3, 3)$ and that the solution for any such system of four independent linear equations in four variables whose coefficients are *cubes* of numbers in arithmetic progression is the 4-tuple $(-1, 4, -6, 4)$, one's interest immediately picks up—$(1, 2)$, $(1, 3, 3)$, and $(1, 4, 6, 4)$ are binomial coefficients. Aha! Indeed, since $(1, -3, 3)$ and $(-1, 4, -6, 4)$ are the solutions for any such systems, we note that

$$a^2 - 3(a + d)^2 + 3(a + 2d)^2 = (a + 3d)^2$$

and

$$-a^3 + 4(a + d)^3 - 6(a + 2d)^3 + 4(a + 3d)^3 = (a + 4d)^3$$

are also identities in a and d. Is this true in general? Is it the case that

$$a^{n-1}x_1 + (a+d)^{n-1}x_2 + \cdots + [a+(n-1)d]^{n-1}x_n = (a+nd)^{n-1} \tag{1}$$

with $(x_1, x_2, \ldots, x_n)$ replaced by

$$(-1)^{n-1}\left(\binom{n}{0}, -\binom{n}{1}, \binom{n}{2}, \ldots, (-1)^{n-1}\binom{n}{n-1}\right) \tag{2}$$

is an identity in a and d?

Comparing the coefficients of a^{n-1} when (1) is expanded, we see that, for (2) to be a solution, we must have

$$\sum_{i=0}^{n-1}(-1)^{n-1+i}\binom{n}{i} = 1.$$

Since $1 = \binom{n}{n}$, this is equivalent to the well-known identity

$$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i} = 0$$

which follows from setting $x = -1$ in the polynomial $f_0(x) = (x+1)^n$ expanded by the binomial theorem. Similarly, considering the terms $a^{n-1-j}d^j$ on each side of (1) for each j, $1 \le j \le n-1$, we must show that

$$\sum_{i=0}^{n-1}(-1)^{n-1+i}\binom{n}{i} i^j = n^j. \tag{3}$$

Again, since $\binom{n}{n} = 1$, this is equivalent to proving that

$$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i} i^j = 0. \tag{4}$$

To accomplish this, we employ a useful device to introduce the powers i^j into the sum. (See [**1**] where the same technique is used in a different context.) We start with $f_0(x) = (x+1)^n$, and proceed to define $f_1(x)$, $f_2(x)$, $\ldots$, $f_{n-1}(x)$ as follows.

$$\begin{aligned}
f_0(x) = (x+1)^n &= \sum_{i=0}^{n}\binom{n}{i} x^i,\\
f_0'(x) = n(x+1)^{n-1} &= \sum_{i=1}^{n}\binom{n}{i} i x^{i-1},\\
f_1(x) = x f_0'(x) = nx(x+1)^{n-1} &= \sum_{i=1}^{n}\binom{n}{i} i x^{i}, \qquad (5)\\
f_1'(x) = n(x+1)^{n-1} + n(n-1)x(1+x)^{n-2} &= \sum_{i=1}^{n}\binom{n}{i} i^2 x^{i-1},\\
f_2(x) = x f_1'(x) = nx(x+1)^{n} + n(n-1)x^2(1+x)^{n-2} &= \sum_{i=0}^{n}\binom{n}{i} i^2 x^{i}
\end{aligned}$$

and so on. In general,

$$f_j(x) = \sum_{i=1}^{n} \binom{n}{i} i^j x^i, \quad 1 \le j \le n-1$$

so that

$$f_j(-1) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^j \tag{6}$$

as desired on the left hand of (4). Moreover, as in (5), $f_j(x)$ consists of terms of the form

$$c_k x^k (x+1)^{n-k}, \quad 1 \le k \le j \le n-1,$$

and it follows that $f_j(-1) = 0$. Therefore, from (6),

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} i^j = 0$$

as required, and the proof is complete.

Comment

One should always leave a problem open-ended; it is never known where an investigation might lead. In this case, we were lead to the interesting indentity in (4); a result already known to those familiar with Stirling numbers of the second kind. In general, students will not be familiar with these numbers and they should be encouraged to ask, "Can the result of (4) be extended?" In particular, we challenge readers not already familiar with them to guess and prove the remarkable results obtained by consdering the sum in (4) for $j = n$ and $n + 1$.

References

1. H. A. Krishnapriyan, Eulerian polynomials and sums of powers of integers, *This Journal*, 26:2 (1995), 118–123.
2. C. T. Long, G. L. Cohen, T. Langtry, and A. G. Shannon, Arithmetic sequences and second order recurrences, in *Applications of Fibonacci Numbers, Vol. 5*, Kluwer Academic Publishers, 1993.
3. C. T. Long, and W. A. Webb, Fundamental solutions of $u^2 - 5v^2 = -4r^2$, in *Applications of Fibonacci Numbers, Vol. 7*, Kluwer Academic Publishers, 1998.