Mathematical Modeling

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Introduction

> *"The muscles of mathematics are connected to the bones of experimental science by the tendons of mathematical modeling."* Glenn Ledder [3].

The pursuit of abstract mathematical knowledge for its own sake holds a venerable and well-deserved position among the activities worthy of an educated person. However, as suggested by Ledder's metaphor in the preface to his textbook on mathematical applications to biology, the field of mathematics connects to the full richness of human experience through the process of mathematical modeling. Much of the relevance and value to be found in the study of applied mathematics in the 21st Century follows from its ability to capture the structure of observable experiences and, consequently, to support our efforts to solve a wide array of real and meaningful problems.

Because mathematical modeling plays a vital role in delivering the power of mathematics to the needs of science, commerce, politics, and so many other areas of human interest, undergraduate programs in mathematics should seek to provide intentional, substantive learning opportunities for their students in the experience of mathematical modeling. However, the manner in which these experiences are situated in the curriculum may vary considerably, depending on the mission, size, resources, and setting of each institution. This report identifies considerations that would be common to the teaching and learning of mathematical modeling regardless of implementation, outlines several viable ways to situate mathematical modeling within the undergraduate program (not all of which entail stand-alone courses in modeling), and provides resources to assist with the development of modeling curricula.

Definition of Mathematical Modeling

Mathematical modeling is best understood as an active process, rather than a static object of study. In practice, modeling entails a systematic approach to problem solving that brings the techniques and structures of mathematics to bear in an effort to describe, understand, and make predictions about a wide range of empirical phenomena. Genuine mathematical modeling is dynamic and iterative in a way that isolated "story problems" or brief applications (often included as short examples near the end of each section in a mathematics textbook) are not. A recent report from a joint SIAM-NSF workshop drew a similar distinction between "mathematical modeling" and "mathematical models":

> Mathematical modeling is an abstract and/or computational approach to the scientific method, where hypotheses are made in the form of mathematical statements (or mathematical models), which are then used to make predictions and/or decisions. The quality of these models is then examined as part of the verification process, and the entire cycle repeats as improvements and adjustments to the model are made. The teaching of models, by contrast, is simply a presentation of the final product, and does not provide many insights into the process or the understanding gleaned from it….It's like the difference between painting a picture and looking at paintings in a museum. [4]

Although there is surely something to be gained by introducing students to significant and interesting mathematical *models* at points throughout the undergraduate mathematics curriculum, this report is primarily concerned with the teaching and learning of *modeling* in the undergraduate mathematics curriculum. By definition and design this should entail an iterative process in which students establish assumptions, develop a mathematical structure consistent with those assumptions, produce hypotheses or conjectures that are generated by the mathematical structure, test the hypotheses against empirical evidence, and then revise/refine the model accordingly.

The physical sciences and engineering fields have long served as the epicenter for mathematical modeling in the undergraduate curriculum. This is not at all surprising in light of the central role that mathematical language has played in these fields, dating at least as far back as the scientific revolution. However, in the past century--and even in just the last few decades--many other disciplines have emerged as fertile ground for mathematical modeling. These areas include biology (e.g., bioinformatics, ecological studies), medicine (e.g., epidemiology, medical imaging), information science (e.g., neural networks, information assurance), sociology (e.g., social dynamics, social networks, dating/matchmaking services), political science (e.g., apportionment, social choice), business operations (e.g. operations research, resource optimization), economics (e.g., game theory, forecasting, market equilibrium), and finance (e.g., option pricing, portfolio optimization).

While many classical models are continuous in nature (often developed in the context of differential equations), it should be noted that discrete, stochastic, and data-driven methods provide the mathematical infrastructure for many significant applications of mathematics as well. A course that is designed to provide students with a general introduction to modeling should consider integrating methods from several/all of these areas.

Audience and Prerequisites

Traditionally, a modeling course might appear as a junior/senior level offering that would be taken only after calculus and differential equations. When positioned as an upper-division offering, a Mathematical Modeling course is able to leverage a more extensive toolbox of prerequisite skills. Indeed, an upper-division modeling course can serve as an excellent vehicle for recalling and strengthening students' working knowledge of the mathematics they learn throughout the core curriculum. However, there is no intrinsic reason why meaningful experiences with modeling could not occur at virtually any place in the undergraduate mathematics curriculum. In fact, the latest standards for K-12 mathematics, the Common Core

State Standards, call for students to develop proficiency in mathematical modeling, which will increase the number of undergraduates that arrive at universities with some modeling experience [2].

The United States Military Academy, for example, features an integrated sequence of four mathematics courses (Discrete Dynamical Systems and Introduction to Calculus, Calculus I, Calculus II, and Probability & Statistics), throughout which students engage in mathematical modeling [1]. Significant modeling experiences are often presented in the form of Interdisciplinary Lively Application Projects (ILAPs), many of which are available in a volume published by the MAA.

Macalester College serves as an example of a relatively small liberal arts college that offers a modeling-intensive track (within the Mathematics Major) in Applied Mathematics and Statistics. Students in the Macalester Mathematics Program are introduced to mathematical modeling early, often in their first-year courses, and an orientation toward applications remains present in many of the courses that follow. Students participating in the Applied Mathematics and Statistics Track culminate their undergraduate experience with capstone courses in Mathematical Modeling, Continuous Applied Mathematics, Discrete Applied Mathematics, Bayesian Statistics, and/or Survival Analysis.

Harvey Mudd College offers a capstone project through their Mathematics Clinic, where teams of 3 to 5 students work as consultants to solve a problem for a client under the direction of a faculty advisor. Towson University also has students solve problems for clients through their Applied Mathematics Laboratory. Clients pay a fee to cover costs such as equipment and faculty time. Students meet periodically with the clients to ensure that the work is satisfying the clients' needs. Past problems have come from industry and government agencies.

Some undergraduate programs have chosen to focus one or more courses in their curriculum on a particular area of application (e.g. mathematical biology, operations research, engineering problems), either in addition to or in lieu of a broad-based course in mathematical modeling. If designed well, such courses can still introduce students to a good variety of modeling problems while also encouraging participation from students--and perhaps faculty--from multiple disciplines.

While less common, a first-year course in mathematical modeling that incorporates just-in-time techniques from calculus, discrete mathematics, and/or statistics could serve as a pathway to recruit new majors and also provide "existing" first-year math majors with a richer sense of purpose in their undergraduate studies.

Core Learning Objectives

A primary purpose of any modeling course should be to develop students' capacity to solve problems through the use of mathematical models *as a transferable process* that will equip them to address novel problems in the future. While specific objectives and cognitive learning goals may vary with the context of each course, a few relatively universal aims emerge in accord with the essential goal of teaching modeling as a transferable process.

Course Objectives. Courses in mathematical modeling should typically strive to:

1. introduce students to the elements of the mathematical modeling process;
2. present application-driven mathematics motivated by problems from within and outside mathematics;
3. exemplify the value of mathematics in problem solving; and
4. demonstrate connections among different mathematical topics.

Effectively modeling courses should seek student learning outcomes that are consistent with the following learning goals.

Cognitive Learning Goals. Students who successfully complete a course in mathematical modeling should be able to:

1. translate everyday situations into mathematical statements (models) which can be solved/analyzed, validated, and interpreted in context;
2. identify assumptions which are consistent with the context of the problem and which in turn shape and define the mathematical characterization of the problem;
3. revise and improve mathematical models so that they will better correspond to empirical information and/or will support more realistic assumptions;
4. assess the validity and accuracy of their approach relative to what the problem requires;
5. work as members of a team toward a common goal, and
6. communicate mathematics in both oral and written form to a broad mathematical and lay audience, including the "end users" of a modeling problem, who may be utterly unfamiliar with the mathematics used.

Pedagogy

Courses focused on mathematical modeling are particularly well suited to a teaching and learning environment that is centered around open-ended inquiry and experimentation. By its very nature, the modeling process admits many possible "solutions" to a given problem, a condition that encourages students to critically evaluate alternative approaches and to attempt multiple paths toward the intended goal. Approximations and estimates are also allowable solutions and the required accuracy of the approximations must be part of the modeling process. Students may even benefit from opportunities to compete for the "best" solution to a modeling problem and to present arguments supporting the model they have developed. (This dynamic has been cultivated at a larger scale with considerable success through modeling contests such as the Mathematical Contest in Modeling (MCM) and the SIAM Moody's Math Challenge.)

When the problems presented to students originate from real circumstances in the surrounding community, mathematical modeling experiences also provide opportunities for students to engage with real clients, in much the same way as they would as professionals in the field. Depending on the scope and timing afforded by the academic calendar, students may have an opportunity to consult with clients to assess their current circumstances and needs, to present preliminary versions of a solution for feedback and revision, to present a "final" version for implementation, and possibly even to observe the effectiveness of the model when implemented.

With these general considerations in mind, the proceeding section lists some suggested approaches to teaching a course that features mathematical modeling.

Tips for Modeling in the Classroom

The *process* of mathematical modeling is best learned in the context of actual modeling problems, not as a separate topic in the syllabus. In a typical stand-alone course, students should be given the opportunity to explore many different modeling scenarios. This section contains advice on how to implement mathematical modeling in a stand-alone course or as part of a course.

Use a problem-oriented approach: It is valuable for students to read an authentic open-ended problem and suggest answers to questions like: "What information do we need to address the problem?" "Is the needed information available somewhere?" "Are there assumptions we should make?" "What mathematics may be useful to address the problem?" Once the students have some of these answers, they can begin to formulate a model based on mathematics they know.

Build on students' ideas: Once students formulate models and use them to answer questions, the doors are opened to discussions of concepts that some students offered but were unfamiliar to others. New material relevant to the problem may be introduced at this point. Some problems might be revisited after students have learned new material that could suggest alternative models for the same situation.

Work in teams: Teamwork promotes communication and cross-fertilization of ideas, mirrors the environment in which this work is typically carried out in the field, and fosters an enthusiastic atmosphere that will make the course more interesting for everyone involved.

Let students be creative: Because modeling is a creative process, it is a good idea to encourage creativity in their assumptions and models. This also promotes developing criteria for validating model conclusions and emphasizes the cyclic nature of the process (where models are revised and improved incrementally). Here it is also important not to steer students into a predetermined model that the instructor may have in mind. Allow students enough time to grapple with the problem.

Describe the stages of modeling: Once students have experienced the modeling process through examples, it is a good idea to systematically go over the modeling cycle and the elements that each stage might entail.

Let students write: Modeling projects, especially those that culminate with some sort of "deliverable," require writing in a way that many undergraduate mathematics courses do not. These situations typically call for a careful mix of technical language along with descriptive--or in some cases persuasive--language which addresses the needs of the audience or end user. Students may benefit from examining sample papers from the Mathematical Contest in Modeling (MCM), which have been published in the UMAP Journal. (A reference and link have been provided in the "Other Resources" section of this report.)

Let students present: Presentations of modeling results are a good way for students to learn to communicate mathematical concepts, to justify choices made in the model, and to critique other points of view.

Choose situations to model from multiple sources: Traditional modeling situations tend to come from science (astronomy, physics, ecology, etc.) but students are also interested in other sources like truth in advertisement (claims made in ads), social networks, political science, economics, health, local communities, sports, etc. The more students relate to the topic, the more engaged they will be.

A faculty member may feel more comfortable introducing mathematical models and then building up to the process of mathematical modeling once some specific models have been evaluated. This can be successful when a model is introduced and then evaluated, before a second model is introduced for the same problem. Highlighting deficiencies or inaccuracies of a first model, and how a second model addresses these aspects, follows and reinforces the steps of the modeling process. When modeling is a component of another class, the modeling process can be used to motivate the just-in-time introduction of new material to alleviate shortcomings of an initial model.

Technology

While there is some pedagogical value in performing certain calculations by hand, an effective modeling course will almost certainly reflect the important role that technology plays as tools for visualization, simulation, and computation in the actual practice of mathematical modeling. Students who learn to make productive and appropriate use of technology in the modeling process not only will be able to address more interesting and sophisticated problems within the course, but also will develop essential skills used by professionals in the field.

Many mathematics programs already incorporate software such as Maple, *Mathematica*, Matlab, R, or Sage across multiple courses in their curriculum. If so, a modeling course may be an excellent place to introduce, leverage, and/or reinforce facility with software that will be utilized elsewhere in the program.

Types of Modeling Problems

When conducted effectively, the teaching and learning of modeling is a problem-driven enterprise. In such a setting, the student is presented with *problems* that can be addressed through effective modeling, rather than being presented with *models*. This environment not only lends itself to a richer learning experience (as it demands higher-order cognitive activities than merely “learning about models”), but also more closely approximates the actual practice of modeling, as it is experienced outside the academic classroom.

The categories of *discrete*, *continuous*, *stochastic*, and *data-driven* (or *statistical*) are commonly used as classifications for types of modeling problems. In introductory undergraduate courses, discrete models may entail methods from such areas as graph theory, scheduling, and linear programming, among others. Continuous modeling problems may draw on techniques from fields including differential equations and dynamical systems. Stochastic models, which may involve either discrete or continuous structures, are characterized by probabilistic descriptions of future states. Markov processes and queuing theory are among the stochastic methods undergraduate students might use in their models. Statistical modeling may entail techniques such as regression analysis, cluster analysis, or hypothesis testing.

Finding and Choosing Modeling Problems

Although aimed at a younger audience, SIAM's Moody's Math Challenge has examples of excellent modeling problems for the advanced high school junior or senior. These problems could easily be made more sophisticated for an undergraduate audience and can serve as a great modeling starting point. The archive that contains all of the past problems can be found at the link below.

http://m3challenge.siam.org/about/archives/

The Mathematical Contest in Modeling (MCM) is a contest where teams of undergraduate students solve real world problems using mathematical techniques. The teams are challenged to tackle an open ended problem from beginning to end and craft an elegant summary of their result. Past problems can be found in the archive below and solutions are published in the UMAP Journal.

http://www.comap.com/undergraduate/contests/matrix/index.html

The archive of problems from Moody's Math Challenge and MCM have been vetted by teachers, industry professionals and professors and are an excellent resource for instructors seeking examples of large scale mathematical modeling problems.

Modeling problems can be divided, at least roughly, into "strong" and "weak" varieties. Strong modeling problems can be defined as those that are authentic (not contrived as a mere classroom exercise), admit many points of entry (e.g. allow different types of models based on different assumptions), and permit the development of simplified models initially and improved models iteratively. Weak problems have an "obvious" way of being modeled that students tend to get forced into, which inhibits creativity. Take for example the following two problems:

Problem 1: *Assume that a wolf population is such that (1) the population becomes extinct if the number of wolf falls below a minimum survival level;(2) if the population is above the minimum survival level, then the growth is limited by the carrying capacity; and (3) if the population is above the carrying capacity it will decline because the environment cannot sustain it. What is a model for this situation?*

Problem 2: *How much daylight does Tucson get in one year?*

The first problem has several shortcomings as a modeling problem. First, it uses language like "minimum survival level" and "carrying capacity," which are normally used in ordinary differential equation models of logistic type. This means that the problem strongly suggests what model to use. It removes the opportunity for the student to choose relevant mathematics and essential variables to build his or her model. The problem itself contains the variables that should be accounted for. On the other hand, problem 2 is completely open-ended. There is no suggestion of how to go about modeling the situation. In fact, the students would have to interpret the question and ask themselves what information and assumptions would be required to state a model. Once they decide, this information must be researched and found. Different models (deterministic or statistical) are possible and the model conclusions are open to interpretation and validation.

Special Considerations

Modeling Across the Curriculum versus a Stand-Alone Modeling Course

Stand-alone courses in mathematical modeling are well-positioned to provide the time, depth, and focus required for authentic modeling experiences. For this reason, we recommend that undergraduate programs in mathematics provide one or more courses for which the process of mathematical modeling is the primary focus.

However, constraints such as limited staffing resources or modest enrollments may render a dedicated course in mathematical modeling infeasible for some programs. Under such circumstances, a curriculum that provides meaningful contact with mathematical modeling across many of its existing courses can be an effective solution, possibly as an intermediate step while the program cultivates the capacity necessary to support a stand-alone course in mathematical modeling. To support the integration of modeling projects and experiences across the undergraduate mathematics curriculum, this document includes a list of resources (detailed in the "Other Resources" section of this report) that provide instructors with ideas, modules, and sample problems suitable for use in core mathematics courses, such as calculus, differential equations, linear algebra, and discrete mathematics. Instructors for these courses may also wish to adopt textbooks that provide an emphasis on modeling, beyond the obligatory example or exercises at the end of each section.

Indeed, because the integration of interesting applications and nontrivial modeling experiences would likely add relevance and richness to most undergraduate mathematics course, we recommend that all programs seek to build modeling components into their existing courses, including those programs that offer stand-alone courses in mathematical modeling.

Meeting the Needs of Pre-service Teachers

Mathematical Practice 4 (MP4) from the Common Core State Standards - Mathematics includes the following language:

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace...Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. [2]

This language reflects an emphasis on skills and perspectives associated with mathematical modeling as an essential component of the K-12 mathematics curriculum. For this reason, colleges and universities that prepare future teachers will want to ensure that their mathematics curriculum is configured in such a way that pre-service teachers gain meaningful experience with modeling and applications of mathematics, either through modeling activities integrated into the core mathematics curriculum, through a dedicated course focused on mathematical modeling, or both.

References

[1] Arney, David, and Snook, Kathleen, "Mathematical Modeling in the USMA Curriculum," October 3, 2014.

[2] Common Core State Standards Initiative, "Standards for Mathematical Practice", 2014.

[3] Ledder, Glenn, *Mathematics for the Life Sciences*, Springer, 2013.

[4] Society for Industrial and Applied Mathematics, "Modeling across the Curriculum," Report from a Joint SIAM-NSF Workshop, Arlington, VA, August 30-31, 2012.

Textbooks:

> Remark*: The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of Mathematical Modeling courses.*

1. Adler, F., Modeling the Dynamics of Life: Calculus and Probability for Life Scientists, Cengage, 2011.

   Introduces/reviews concepts and techniques from calculus, probability and statistics with an emphasis on modeling biological phenomena. Suitable for biology majors as well as mathematics majors.

2. Giordano, F. R., Fox, W. P., Horton, S. B., and Weir, M. D., *A First Course in Mathematical Modeling*, 4th ed., Cengage Learning, 2013.

   Comprehensive introduction to the modeling process, including model construction, simulation, and model analysis. Includes discrete, continuous, stochastic, and data-driven methods. Early chapters assume minimal prerequisites. Some later chapters assume at least first-year calculus.

3. Hadlock, C., *Mathematical Modeling in the Environment*, MAA, 1998.

   Emphasizes the interdisciplinary nature of mathematical modeling, within the context of issues in environmental science. The book begins with algebra-based models. Later chapters include models that draw on more sophisticated mathematical techniques,

including multivariable calculus, differential equations, numerical analysis and linear algebra.

4. Heinz, S., *Mathematical Modeling*, Springer-Verlag, 2011.

   Appropriate for a more sophisticated course in mathematical modeling for advanced undergraduate students. Applications are drawn from a wide range of fields in the physical sciences, biology, and economics.

5. Mooney, D., and Swift, R., *A First Course in Mathematical Modeling*, MAA, 1999.

   Incorporates discrete, continuous, stochastic, and data-driven approaches, largely directed toward applications from biology and ecology. Computer assisted analysis and simulation is widely used. Assumes first-year calculus as a prerequisite.

6. Olinick, M., *Mathematical Modeling in the Social and Life Sciences*, Wiley, 2014.

   Features a broad range of applications from the social sciences, life sciences, and humanities. Includes discrete, continuous, and stochastic approaches. Suitable for a freshman or sophomore level course in mathematical modeling.

7. Roberts, Fred S., *Discrete Mathematical Models with Applications to Social, Biological, and Environmental Problems*, 1st Ed., Pearson, 1976.

   Applications of graph theory, Markov chains, game theory, and social choice theory to the social, biological, and environmental sciences. Text may be used in courses from first year to senior level.

8. Shiflet, A.B., and Shiflet, G.W., *Introduction to Computational Science: Modeling and Simulation for the Sciences*, 2nd Ed., Princeton, 2014.

   Accessible to first-year students. Uses modeling in computational science as the vehicle for studying problems across the physical and natural sciences.

Other resources:

9. Arney, D., *Interdisciplinary Lively Application Projects*, MAA Classroom Resource Materials, 1997.

10. Consortium for Mathematics and its Applications (COMAP), Mathematical Contest in Modeling (MCM)/Interdisciplinary Contest in Modeling (ICM) Website.

    Site contains past problems and supplemental resources associated with COMAP's undergraduate modeling contests.

11. Fraga, R., *War Stories from Applied Math: Undergraduate Consultancy Projects*, MAA Notes, 2006.

12. Society for Industrial and Applied Mathematics (SIAM), Moody's Mega Math Challenge Website.

Site contains past problems and supplemental resources associated with the Moody's Mega Math Challenge high school modeling contest. Many of the problems and ideas can be adapted to an undergraduate audience.