A Menelaus-Type Theorem for the Pentagram

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A pentagram, or star-polygon, is formed by the diagonals of a convex pentagon such as that shown in Figure 1. The regular pentagram has been studied extensively beginning with the Pythagoreans who used it as an emblem of their society. The most ubiquitous property of the regular pentagram is that involving the golden ratio [1]. A very brief history of regular pentagrams is given in [3, pp. 44–45]. In this note we derive some additional properties for arbitrary pentagrams.

Our main result is reminiscent of the theorems of Menelaus and Ceva in that the products of ratios of segments taken in order around a polygon is equal to $\pm 1$. More precisely:

**Theorem 1.** If $A_1B_1A_2B_2A_3B_3A_4B_4A_5B_5$ is a pentagram, then

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot \frac{A_4B_4}{B_4A_5} \cdot \frac{A_5B_5}{B_5A_1} = 1.$$  

For the proof we repeatedly use Menelaus’ Theorem, which states that if a line intersects the (extended) sides of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of $\triangle ABC$ in points $D, E, F$, respectively, then

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1.$$
Proofs of this famous theorem can be found in many geometry books—one example is [2].

For our proof we find it convenient to let \( A_{i+5} = A_i \) and \( B_{i+5} = B_i \) for \( i = 1, 2, \ldots, 5 \). We use Menelaus’ Theorem in blocks of five triangles as follows:

For triangles of the form \( A_i B_i B_{i+4} \) and lines \( \overline{A_{i+1}A_{i+3}} \), we have

\[
\frac{A_1 B_2}{B_2 A_1} \cdot \frac{B_1 A_2}{A_2 B_5} \cdot \frac{B_5 A_4}{A_4 A_1} = -1, \\
\frac{A_2 B_3}{B_3 A_2} \cdot \frac{B_2 A_3}{A_3 B_1} \cdot \frac{B_1 A_5}{A_5 A_2} = -1, \\
\frac{A_3 B_4}{B_4 B_3} \cdot \frac{B_3 A_4}{A_4 B_2} \cdot \frac{B_2 A_1}{A_1 A_3} = -1, \\
\frac{A_4 B_5}{B_5 A_4} \cdot \frac{B_4 A_5}{A_5 B_3} \cdot \frac{B_3 A_2}{A_2 A_4} = -1 \text{ and} \\
\frac{A_5 B_1}{B_1 B_5} \cdot \frac{B_5 A_1}{A_1 B_4} \cdot \frac{B_4 A_3}{A_3 A_5} = -1. 
\]

(1)

For the same triangles \( A_i B_i B_{i+4} \) but for lines \( \overline{A_{i+2}A_{i+4}} \), we have

\[
\frac{A_1 A_3}{A_3 B_1} \cdot \frac{B_1 A_5}{A_5 B_5} \cdot \frac{B_5 B_1}{B_1 A_1} = -1, \\
\frac{A_2 A_4}{A_4 B_2} \cdot \frac{B_2 A_1}{A_1 B_1} \cdot \frac{B_1 B_5}{B_5 A_2} = -1, \\
\frac{A_3 A_5}{A_5 B_3} \cdot \frac{B_3 A_2}{A_2 B_2} \cdot \frac{B_2 B_1}{B_1 A_3} = -1, \\
\frac{A_4 A_1}{A_1 B_4} \cdot \frac{B_4 A_3}{A_3 B_3} \cdot \frac{B_3 B_2}{B_2 A_4} = -1 \text{ and} \\
\frac{A_5 A_2}{A_2 B_5} \cdot \frac{B_5 A_4}{A_4 B_4} \cdot \frac{B_4 B_3}{B_3 A_5} = -1. 
\]

(2)

By multiplying together the ten equations of (1) and (2), simplifying, and rearranging factors we have

\[
\left[ \frac{A_1 B_2}{B_2 A_4} \cdot \frac{A_4 B_3}{B_5 A_2} \cdot \frac{A_2 B_3}{B_3 A_5} \cdot \frac{A_5 B_1}{B_1 A_3} \cdot \frac{A_3 B_4}{B_4 A_1} \right]^3 \times \left[ \frac{B_5 A_1}{A_1 B_1} \cdot \frac{B_1 A_2}{A_2 B_2} \cdot \frac{B_2 A_3}{A_3 B_3} \cdot \frac{B_3 A_4}{A_4 B_4} \cdot \frac{B_4 A_5}{A_5 B_5} \right] = 1.
\]

(3)

Similarly for triangles of the form \( A_i A_{i+2} B_{i+3} \) and lines \( \overline{A_{i+1}A_{i+4}} \), we have

\[
\frac{A_1 B_1}{B_1 A_3} \cdot \frac{A_3 A_5}{A_5 B_4} \cdot \frac{B_5 B_1}{B_1 A_1} = -1, \\
\frac{A_2 B_2}{B_2 A_4} \cdot \frac{A_4 A_1}{A_1 B_5} \cdot \frac{B_5 B_1}{B_1 A_2} = -1, \\
\frac{A_3 B_3}{B_3 A_5} \cdot \frac{A_5 A_2}{A_2 B_1} \cdot \frac{B_1 B_2}{B_2 A_3} = -1. 
\]

(4)
For the same triangles $A_iA_{i+1}B_{i+3}$ but for lines $A_iA_{i+1}A_{i+3}$, we have

\[
\frac{A_1B_2}{B_2A_3} \cdot \frac{A_2B_3}{B_3A_4} \cdot \frac{A_3B_4}{B_4A_5} = -1 \quad \text{and} \\
\frac{A_1B_4}{B_4A_1} \cdot \frac{A_2B_5}{B_5A_2} \cdot \frac{A_3B_6}{B_6A_3} = -1.
\]

By multiplying the ten equations of (4) and (5), simplifying, and rearranging the factors, we obtain

\[
\left[ \frac{A_1B_2}{B_2A_3} \cdot \frac{A_4B_5}{B_5A_2} \cdot \frac{A_2B_3}{B_3A_5} \cdot \frac{A_5B_1}{B_1A_3} \cdot \frac{A_3B_4}{B_4A_1} \right]^3 \times \left[ \frac{A_1B_4}{B_4A_1} \cdot \frac{A_2B_5}{B_5A_2} \cdot \frac{A_3B_6}{B_6A_3} \cdot \frac{A_4B_7}{B_7A_4} \cdot \frac{A_5B_8}{B_8A_5} \right] = 1.
\]

By dividing equation (3) by (6), observing that the cubed factors are identical, and the second factors are reciprocals, we obtain

\[
\left[ \frac{A_1B_2}{B_1A_2} \cdot \frac{A_2B_3}{B_2A_3} \cdot \frac{A_3B_4}{B_3A_4} \cdot \frac{A_4B_5}{B_4A_5} \right]^3 = 1
\]

and the result follows. As a bonus of (3) and (6) we note that

\[
\frac{A_1B_2}{B_2A_4} \cdot \frac{A_4B_5}{B_5A_2} \cdot \frac{A_2B_3}{B_3A_5} \cdot \frac{A_5B_1}{B_1A_3} \cdot \frac{A_3B_4}{B_4A_1} = 1.
\]

REFERENCES