A Menelaus-Type Theorem for the Pentagram

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A pentagram, or star-polygon, is formed by the diagonals of a convex pentagon such as that shown in FIGURE 1. The *regular* pentagram has been studied extensively beginning with the Pythagoreans who used it as an emblem of their society. The most ubiquitous property of the regular pentagram is that involving the golden ratio [**1**]. A very brief history of regular pentagrams is given in [**3**, pp. 44–45]. In this note we derive some additional properties for arbitrary pentagrams.

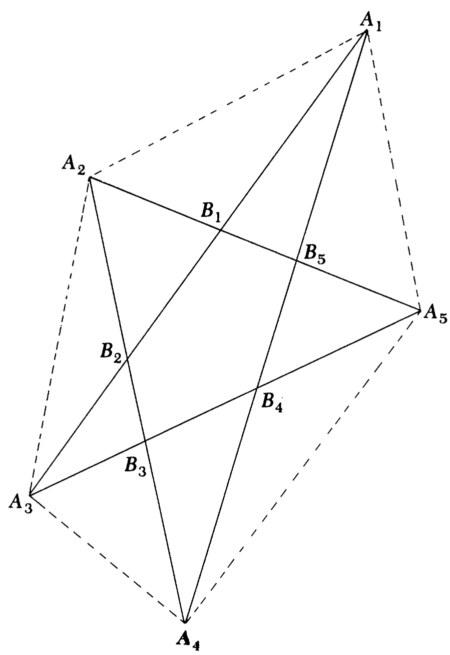

FIGURE 1

Our main result is reminiscent of the theorems of Menelaus and Ceva in that the products of ratios of segments taken in order around a polygon is equal to ± 1. More precisely:

THEOREM 1. *If* $A_1B_1A_2B_2A_3B_3A_4B_4A_5B_5$ *is a pentagram, then*

$$\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}=1.$$

For the proof we repeatedly use Menelaus' Theorem, which states that if a line intersects the (extended) sides of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of ΔABC in points D, E, F, respectively, then

$$\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF}{FA}=-1.$$

Proofs of this famous theorem can be found in many geometry books—one example is [2].

For our proof we find it convenient to let $A_{i+5} = A_i$ and $B_{i+5} = B_i$ for $i = 1, 2, \dots, 5$. We use Menelaus' Theorem in blocks of five triangles as follows:

For triangles of the form $A_iB_iB_{i+4}$ and lines $\overline{A_{i+1}A_{i+3}}$, we have

$$\begin{aligned} \frac{A_1B_2}{B_2B_1}\cdot\frac{B_1A_2}{A_2B_5}\cdot\frac{B_5A_4}{A_4A_1} &= -1,\\ \frac{A_2B_3}{B_3B_2}\cdot\frac{B_2A_3}{A_3B_1}\cdot\frac{B_1A_5}{A_5A_2} &= -1,\\ \frac{A_3B_4}{B_4B_3}\cdot\frac{B_3A_4}{A_4B_2}\cdot\frac{B_2A_1}{A_1A_3} &= -1,\\ \frac{A_4B_5}{B_5B_4}\cdot\frac{B_4A_5}{A_5B_3}\cdot\frac{B_3A_2}{A_2A_4} &= -1 \text{ and}\\ \frac{A_5B_1}{B_1B_5}\cdot\frac{B_5A_1}{A_1B_4}\cdot\frac{B_4A_3}{A_3A_5} &= -1. \end{aligned} \tag{1}$$

For the same triangles $A_iB_iB_{i+4}$ but for lines $\overline{A_{i+2}A_{i+4}}$, we have

$$\begin{aligned} \frac{A_1A_3}{A_3B_1}\cdot\frac{B_1A_5}{A_5B_5}\cdot\frac{B_5B_4}{B_4A_1} &= -1,\\ \frac{A_2A_4}{A_4B_2}\cdot\frac{B_2A_1}{A_1B_1}\cdot\frac{B_1B_5}{B_5A_2} &= -1,\\ \frac{A_3A_5}{A_5B_3}\cdot\frac{B_3A_2}{A_2B_2}\cdot\frac{B_2B_1}{B_1A_3} &= -1,\\ \frac{A_4A_1}{A_1B_4}\cdot\frac{B_4A_3}{A_3B_3}\cdot\frac{B_3B_2}{B_2A_4} &= -1 \text{ and}\\ \frac{A_5A_2}{A_2B_5}\cdot\frac{B_5A_4}{A_4B_4}\cdot\frac{B_4B_3}{B_3A_5} &= -1. \end{aligned} \tag{2}$$

By multiplying together the ten equations of (1) and (2), simplifying, and rearranging factors we have

$$\left[\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}\right]^3 \times \left[\frac{B_5A_1}{A_1B_1}\cdot\frac{B_1A_2}{A_2B_2}\cdot\frac{B_2A_3}{A_3B_3}\cdot\frac{B_3A_4}{A_4B_4}\cdot\frac{B_4A_5}{A_5B_5}\right] = 1. \tag{3}$$

Similarly for triangles of the form $A_iA_{i+2}B_{i+3}$ and lines $\overline{A_{i+1}A_{i+4}}$, we have

$$\begin{aligned} \frac{A_1B_1}{B_1A_3}\cdot\frac{A_3A_5}{A_5B_4}\cdot\frac{B_4B_5}{B_5A_1} &= -1,\\ \frac{A_2B_2}{B_2A_4}\cdot\frac{A_4A_1}{A_1B_5}\cdot\frac{B_5B_1}{B_1A_2} &= -1,\\ \frac{A_3B_3}{B_3A_5}\cdot\frac{A_5A_2}{A_2B_1}\cdot\frac{B_1B_2}{B_2A_3} &= -1, \end{aligned} \tag{4}$$

$$\frac{A_4B_4}{B_4A_1}\cdot\frac{A_1A_3}{A_3B_2}\cdot\frac{B_2B_3}{B_3A_4}=-1 \text{ and}$$

$$\frac{A_5B_5}{B_5A_2}\cdot\frac{A_2A_4}{A_4B_3}\cdot\frac{B_3B_4}{B_4A_5}=-1.$$

For the same triangles $A_iA_{i+2}B_{i+3}$ but for lines $\overline{A_{i+1}A_{i+3}}$, we have

$$\begin{aligned}
&\frac{A_1B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3B_4}\cdot\frac{B_4A_4}{A_4A_1}=-1,\\
&\frac{A_2B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4B_5}\cdot\frac{B_5A_5}{A_5A_2}=-1,\\
&\frac{A_3B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5B_1}\cdot\frac{B_1A_1}{A_1A_3}=-1,\\
&\frac{A_4B_5}{B_5A_1}\cdot\frac{A_1B_1}{B_1B_2}\cdot\frac{B_2A_2}{A_2A_4}=-1 \text{ and}\\
&\frac{A_5B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2B_3}\cdot\frac{B_3A_3}{A_3A_5}=-1.
\end{aligned} \tag{5}$$

By multiplying the ten equations of (4) and (5), simplifying, and rearranging the factors, we obtain

$$\left[\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}\right]^3\times\left[\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}\right]=1. \tag{6}$$

By dividing equation (3) by (6), observing that the cubed factors are identical, and the second factors are reciprocals, we obtain

$$\left[\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}\right]^2=1$$

and the result follows. As a bonus of (3) and (6) we note that

$$\frac{A_1B_2}{B_2A_4}\cdot\frac{A_4B_5}{B_5A_2}\cdot\frac{A_2B_3}{B_3A_5}\cdot\frac{A_5B_1}{B_1A_3}\cdot\frac{A_3B_4}{B_4A_1}=1.$$

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3. Howard Eves, *A Survey of Geometry* (rev. ed.), Allyn and Bacon, Inc., Boston, 1972.