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Fooling Newton's Method as Much as One Can

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We enjoyed reading how Horton [**1**] "fooled Newton's method" with an example where the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges but its limit does not satisfy $f(x) = 0$. Indeed, if

$$f(x) = \begin{cases} \pi - 2x \sin \frac{\pi}{x} & \text{for } x \neq 0, \\ \pi & \text{for } x = 0, \end{cases} \tag{1}$$

then the Newton sequence is

$$x_{n+1} = x_n - \frac{1}{2} \frac{\pi x_n - 2x_n^2 \sin \frac{\pi}{x_n}}{\pi \cos \frac{\pi}{x_n} - x_n \sin \frac{\pi}{x_n}},$$

and, starting from $x_1 = 1/2$, we have $x_2 = 1/4, x_3 = 1/8, \dots, x_n = 1/2^n \to 0$, although $f(0) = \pi \neq 0$.

Can *f* be differentiable? Note that the function in (1) is not differentiable at $x = 0$. Since we thought that a differentiable function would fool the method even better, we wanted to know if such a function exists. Simply modifying Horton's function, we found an example that readers might find even more surprising:

$$f(x) = \begin{cases} \pi - x^2 \sin \frac{\pi}{x^2} & \text{for } x \neq 0, \\ \pi & \text{for } x = 0. \end{cases} \tag{2}$$

This function is differentiable and its Newton sequence is

$$x_{n+1} = x_n - \frac{1}{2}\frac{\pi x_n - x_n^3 \sin\frac{\pi}{x_n^2}}{\pi \cos\frac{\pi}{x_n^2} - x_n^2 \sin\frac{\pi}{x_n^2}}.$$

Again, if $x_1 = 1/2$, then $x_n = 1/2^n \to 0$, but $f(0) = \pi \neq 0$.

Can f be continuously differentiable? Because f in (2) is not continuously differentiable at $x = 0$, our next question was: Can we fool Newton's method with a continuously differentiable function?

The answer is negative. More generally, f' cannot be bounded near the limit point x_0. (If f' were continuous, then it would be bounded there.) For, assume that $x_n \to x_0$ and f is continuous at x_0. Since

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

and the left-hand side has limit zero while the numerator of the right-hand side has limit $f(x_0)$, it follows that

$$f(x_0) = 0 \quad \text{or} \quad |f'(x_n)| \to \infty.$$

(Note that the existence of $f'(x_0)$ is not needed. It is enough that $f'(x)$ exists when $x \neq x_0$ is near x_0.)

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Ramanujan's 6–8–10 Equation and Beyond

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Among Ramanujan's many beautiful formulas is the 6–8–10 equation

$$\begin{aligned}
&64[(a+b+c)^6 + (b+c+d)^6 - (c+d+a)^6 \\
&\qquad - (d+a+b)^6 + (a-d)^6 - (b-c)^6] \\
&\quad \times [(a+b+c)^{10} + (b+c+d)^{10} - (c+d+a)^{10} \\
&\qquad - (d+a+b)^{10} + (a-d)^{10} - (b-c)^{10}] \\
&\quad = 45[(a+b+c)^8 + (b+c+d)^8 - (c+d+a)^8 \\
&\qquad - (d+a+b)^8 + (a-d)^8 - (b-c)^8]^2
\end{aligned}$$