rational numbers rather than integers is given by $(x/w, y/w, z/w)$, where x, y, z, w are given by Catalan's formulas.

References

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Simultaneous Triangle Inequalities

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It is a well-known result that the necessary and sufficient conditions that three positive numbers be the lengths of the sides of some triangle are that

$$b+c>a, \qquad c+a>b, \quad a+b>c. \tag{1}$$

Clearly it then follows that

$$(b+c-a)(c+a-b)(a+b-c)>0. \tag{2}$$

Also, it is easy to see that for $a, b, c > 0$, (2) $\Rightarrow$ (1). For at most one of the three factors in (2) can be $\leqslant 0$ and this would violate (2). The latter inequality is also equivalent to

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)>0$$

or, by multiplying out, to

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)>0. \tag{3}$$

By Heron's formula for the area F of a triangle [**1**], (3) is given more compactly as

$$16F^2>0. \tag{3'}$$

As an extension of the above results, one can ask does there exist a polynomial inequality in the n positive numbers $a_1, a_2, \ldots, a_n$ which implies that any three of the numbers are lengths of sides of a triangle. Offhand one would expect that such a polynomial inequality exists and also that its degree is at least of order kn. Surprisingly, there is such a polynomial of degree 4 for all $n>3$. Formally, our result is as follows:

If $a_1, a_2, \ldots, a_n > 0$ for $n \geqslant 3$ and

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)^2>(n-1)\left(a_1^4+a_2^4+\cdots+a_n^4\right), \tag{4}$$

then a_i, a_j, a_k, for all $i \neq j \neq k$, are lengths of sides of a triangle.

Our proof is by induction. First we show that (4) implies that

$$\left(a_2^2+a_3^2+\cdots+a_n^2\right)^2>(n-2)\left(a_2^4+a_3^4+\cdots+a_n^4\right), \tag{5}$$

where the left out term, a_1, is arbitrary. After some elementary algebra involved in completing a square, (4) can be shown to be equivalent to

$$0>\left\{a_1^2-S_2/(n-2)\right\}^2-\left\{S_2^2-(n-2)S_4\right\}\{n-1\}/\{n-2\}^2,$$

where $S_m = a_2^m + a_3^m + \cdots + a_n^m$. Consequently, (4) $\Rightarrow$ (5). Then by induction,

$$\left(a_i^2+a_j^2+a_k^2\right)^2>2\left(a_i^4+a_j^4+a_k^4\right),$$

which corresponds to (3).

Inequality (4) arose in my generalization of problem 1087, *Crux Mathematicorum*, 11(1985) 289, i.e., there exists a regular $(n-1)$-dimensional simplex $A_1A_2\cdots A_n$ of edge length a and a point P in its space such that $PA_i=a_i$, $i=1,2,\ldots,n$, if and only if inequality (4) holds.

Note that by applying Ptolemy's inequality in 3-space to the tetrahedron $P\text{-}A_iA_jA_k$, it follows that a_i, a_j, a_k are lengths of sides of a triangle.

It is to be noted that whereas (3) is a necessary and sufficient condition on three positive numbers to be the lengths of sides of a triangle, (4) is only a sufficient condition that any three of n positive numbers are lengths of sides of a triangle. For example, consider the four numbers $5,5,5$, and 9. Any three of them are lengths of sides of a triangle, but (4) for $n=4$ is not satisfied. As an open problem, find a polynomial inequality which is both a necessary and sufficient condition for the latter property. Finally, as another open problem, find a polynomial inequality on n positive numbers $a_1,a_2,\ldots,a_n$ such that any r of them, with $n>r>3$, are lengths of sides of an r-gon.

Reference

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On the Sum of Consecutive *K*th Powers

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In the early 18th century, James Bernoulli discovered an elegant formula for the sum of consecutive Kth powers:

$$1^K+2^K+\cdots+(n-1)^K=\sum_{i=0}^{K}\binom{K}{i}B_i\frac{n^{K+1-i}}{K+1-i}\qquad(K=1,2,\ldots).\tag{1}$$

Here, $B_0, B_1, B_2,\ldots$ are the so-called *Bernoulli numbers* which arise as coefficients in the power series expansion of $x/(e^x-1)$:

$$\frac{x}{e^x-1}=\sum_{K=0}^{\infty}\frac{B_K}{K!}x^K\quad(\text{valid for } |x|<2\pi).$$

The first few values of B_K are

$$B_0=1,\qquad B_1=-\frac{1}{2},\qquad B_2=\frac{1}{6},$$

$$B_3=0,\qquad B_4=-\frac{1}{30},\qquad B_5=0.$$

It is well known that if K is odd and greater than 1, then $B_K=0$, while if K is even and greater