On Some Symmetric Sets of Unit Vectors

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In this note we start with a given set of symmetric unit vectors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ in a Euclidean space with $\Sigma \mathbf{A}_k = \mathbf{0}$ and consider conditions on real numbers $x_1, x_2, \dots, x_n$ that allow us to conclude that

$$\Sigma x_k \mathbf{A}_k = \mathbf{0} \Rightarrow x_1 = x_2 = \cdots = x_n.$$

Different kinds of "symmetry" will lead to different conclusions.

First we consider a known result concerning symmetric unit length complex numbers in the plane. One natural such set is the nth roots of unity

$$1, \omega, \omega^2, \dots, \omega^{n-1}$$

where $\omega = e^{2\pi i/n}$. Since $\omega^n - 1 = 0$, it follows immediately that

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

A converse result also holds, for if

$$x_0 + x_1\omega + x_2\omega^2 + \cdots + x_{n-1}\omega^{n-1} = 0, \tag{1}$$

where $x_0 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1}$, then $x_0 = x_1 = x_2 = \cdots = x_{n-1}$.

Our proof is indirect, so we assume the x_i's are not all equal. On multiplying (1) by $1 - \omega$, we obtain

$$x_0 - x_{n-1} = (x_0 - x_1)\omega + (x_1 - x_2)\omega^2 + \cdots + (x_{n-2} - x_{n-1})\omega^{n-1}. \tag{2}$$

First we consider the case

$$x_0 = x_1 = \cdots = x_k > x_{k+1} = x_{k+2} = \cdots = x_{n-1}.$$

Here (2) reduces to $x_0 - x_{n-1} = (x_k - x_{k+1})\omega^{k+1} = (x_0 - x_{n-1})\omega^{k+1}$, from which $x_0 = x_{n-1}$. For all other cases, we apply the triangle inequality to

$$|x_0 - x_{n-1}| = |(x_0 - x_1)\omega + (x_1 - x_2)\omega^2 + \cdots + (x_{n-2} - x_{n-1})\omega^{n-1}|,$$

which yields

$$x_0 - x_{n-1} < (x_0 - x_1) + (x_1 - x_2) + \cdots + (x_{n-2} - x_{n-1}) = x_0 - x_{n-1},$$

giving the desired contradiction.

For an extension of the above result, we now consider a set of $n+1$ distinct concurrent unit vectors $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_n$ in E^n that are equally inclined to each other. It is geometrically intuitive that

$$\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n = 0 \tag{2}$$

and for which one can give a number of different proofs. First, as suggested by one of the referees, is to use mathematical induction on the dimension n. Secondly, we can use barycentric coordinates noting that from the centroid

$$\mathbf{G} = (\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n)/(n+1),$$

the volumes spanned by any n of the endpoints of the vectors $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ are all equal and the same is true from the origin of the vectors. Thirdly, as suggested by my colleague Jim Pounder, the centroid is unique and if it did not coincide with the origin, then by applying the group of transformations that take the given set of vectors into itself, we would generate many different centroids. Finally, we give an easy self-contained analytic proof based on linear independence. Since the given space is of dimension n, there exist sets of n linearly independent vectors. Any set of n of the vectors $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ are linearly independent since the volumes spanned by each set are equal and > 0. Hence, there is a representation

$$\mathbf{A}_0 = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n. \tag{3}$$

We now take the scalar product of (3) with each of $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$, giving

$$1 = kS, \tag{4}$$

$$k = x_i + (S - x_i)k, \qquad i = 1, 2, \ldots, n \tag{5}$$

where $k = \mathbf{A}_i \cdot \mathbf{A}_j$ $(i \neq j)$ and $S = \Sigma x_i$. Summing (5) over i, gives

$$nk = S + nkS - kS.$$

Now replacing S by $1/k$, we get $(k-1)(nk+1) = 0$. Since the $\mathbf{A}_i$'s are distinct, $k \neq 1$ so that $k = -1/n$. We now calculate the square of the length of the sum of the $n+1$ vectors, i.e.,

$$(\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n)^2 = n + 1 + 2k\binom{n+1}{2} = 0,$$

which gives the desired result.

It now follows as in our introductory example that if

$$x_0\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n = 0, \tag{6}$$

where here the x_i's are any real numbers, not necessarily monotonic (since we have more symmetry here), then the x_i's are all equal. Just square (5) to give

$$\sum_i x_i^2 - 2\sum_{i<j} x_i x_j/n = 0 = \sum_{i<j} (x_i - x_j)^2/n.$$

It is worth noting that one cannot have a set of more than $n+1$ distinct unit vectors in E^n that are equally inclined to each other. If there were $n+2$ such vectors, then as before the sum of every subset of $n+1$ of the vectors must be zero. This implies that the vectors are not distinct, which contradicts the hypotheses.

For our final example, we leave an open problem. Consider a hypercube in E^n with vertices $(\pm 1, \pm 1, \ldots, \pm 1)$. The set of the 2^n vectors from the origin to the vertices is a symmetric set of vectors with sum zero. It is possible to order this set of vectors $\mathbf{A}_i$ such that if

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_{2n}\mathbf{A}_{2n} = 0,$$

where $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{2n}$, then all the x_i's must be equal. The open problem is to find all such possible orderings. In particular, it is immediate that some of the orderings of a Grey Code arrangement are possible, i.e., the number of sign changes in the components between two adjacent vectors including the last with the first is exactly one. For example in E^3, we have the ordering

$$(1,1,1),(1,-1,1),(-1,-1,1),(-1,1,1),(-1,1,-1),$$
$$(-1,-1,-1),(1,-1,-1),(1,1,-1).$$

This ordering works since the third components are in monotonic order. One can also show that the cyclic permutations of these 8 vectors are all possible orderings. Is this also true in E^n?

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A Coordinate Approach to the AM-GM Inequality

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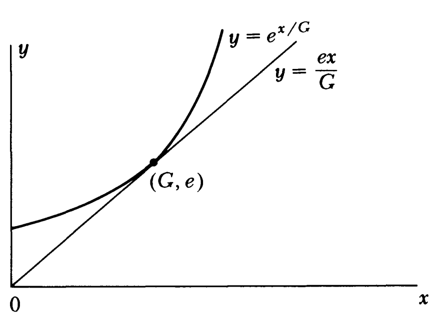
Let $a_1, a_2, \dots, a_n$ be n positive numbers with arithmetic mean A and geometric mean G. The *AM–GM* Inequality states that $A \geqslant G$ with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The graph of $y = e^{x/G}$ is concave upward and thus the tangent line $y = (ex/G)$ at (G, e) lies below the curve. To show $A \geqslant G$, we substitute $x = a_i$ $(i = 1, 2, \dots, n)$ successively into $e^{x/G} \geqslant (ex/G)$ and multiply. Hence

$$e^{(a_1 + a_2 + \cdots + a_n)/G} \geqslant \left(\frac{ea_1}{G}\right)\left(\frac{ea_2}{G}\right) \cdots \left(\frac{ea_n}{G}\right) = e^n.$$

Thus, we have $nA/G \geqslant n$ or $A \geqslant G$, with equality if and only if $a_1 = a_2 = \cdots = a_n = G$.



Acknowledgement. This proof is a geometric variation of the proof suggested by G. L. Alexanderson to Ivan Niven in *Maxima and Minima Without Calculus*, MAA, 1981, pp. 240–241.