$$g_n(t) = \sum_{j=0}^{n} a_j \log_{\kappa_j} t,$$

where $1 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$ and a_j's are real numbers so that $a_n \neq 0$. Changing logarithms to the same base e gives us

$$g_n(t) = \sum_{j=0}^{n} a_j \frac{\ln t}{\ln \kappa_j} = \ln t^{\alpha},$$

where

$$\alpha = \sum_{j=0}^{n} \frac{a_j}{\ln \kappa_j}.$$

Now, depending on whether α is greater than, less than, or equal to zero, g_n is, respectively, strictly increasing with $g_n(1) = 0$, strictly decreasing with $g_n(1) = 0$, or $g_n(t) = 0$ for every $t > 0$.

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Not Mixing Is Just as Cool

SAM NORTHSHIELD
SUNY Plattsburgh
Plattsburgh, NY 12901

Newton's law of cooling is a staple of the Calculus curriculum; it is usually presented as a first or second example of a separable differential equation. In that context, the law states that the rate of change of the temperature T of, say, a quantity of fluid is proportional to the difference between the fluid's temperature and the ambient temperature T_∞:

$$\frac{dT}{dt} = -k(T - T_\infty). \tag{1}$$

This is easily solved (part of the difficulty in solving it is dealing with initial conditions):

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-kt} \tag{2}$$

where $T_0 := T(0)$ is the temperature at time $t = 0$.

The following problem is, for many students, a challenging application of Newton's law even given the formula (2).

PROBLEM. *Which results in a cooler drink:*

(1) *Pour a cup of coffee, wait five minutes, and then add an ounce of cold milk or*

(2) *Pour a cup of coffee, add an ounce of cold milk, and then wait five minutes?*

One challenging aspect of this problem is that a law for the temperature of mixed fluids must be either known *a priori* or else invented during the solution of the problem. We shall give it. If we mix two fluids (that are thermodynamically "similar") then the temperature of the mixed fluid is the average of the temperatures but weighted according to quantity. For example 3 oz. of water at 100 degrees mixed with 2 oz. of water at 150 degrees results in 5 oz. of water at $(3 \cdot 100 + 2 \cdot 150)/5 = 120$ degrees. In general,

PRINCIPLE A. *If Q_A units of fluid at temperature T_A is mixed with Q_B units of fluid at temperature T_B, then the resulting mix has temperature*

$$T := \frac{T_A Q_A + T_B Q_B}{Q_A + Q_B}. \tag{3}$$

We will now solve the problem in two ways, one via Newton's law, the other through intuition.

Solution 1. Let T_c and T_m denote the initial temperatures of the coffee and milk respectively. Let's assume that a cup of coffee is the 'standard' 6 oz. though, as it turns out, this assumption will not affect the answer to the question.

In case 1, after 5 minutes of cooling, equation (2) predicts that the temperature of the coffee is

$$T(5) = T_\infty + (T_c - T_\infty)e^{-5k} \tag{4}$$

and, after mixing with milk, equation (3) predicts a final temperature of $(6T(5) + T_m)/7$ or

$$T_1 := \frac{1}{7}(6T_\infty + T_m) + \frac{6}{7}(T_c - T_\infty)e^{-5k}. \tag{5}$$

In case 2, mixing the coffee and milk yields 7 oz. of fluid at initial temperature $(6T_c + T_m)/7$ which, after cooling for 5 minutes, has temperature

$$T_2 := T_\infty + \frac{1}{7}(6T_c + T_m - 7T_\infty)e^{-5k}. \tag{6}$$

Taking the difference and simplifying, we find

$$T_2 - T_1 = \frac{1}{7}(1 - e^{-5k})(T_\infty - T_m). \tag{7}$$

Hence, assuming that "cold" means colder than the ambient temperature T_∞, Case 1 yields a cooler drink. ■

Note that equation (7) implies that the difference of temperature in Case 1 and Case 2 is independent of the temperature of the coffee. It is, rather, the relative amount of milk that makes the difference. That is, the 7 in equation (7) comes from the ratio 1:6 of milk to coffee. We would replace the 7 by 9 if we used an 8 oz. cup of coffee.

Consider now an 'intuitive' solution:

Solution 2. Suppose we take 1 oz. of milk and allow it to warm up for 5 minutes while simultaneously allowing 6 oz. of coffee to cool. Then we mix them. Since *it*

makes no difference if the fluids were already mixed or not, the temperature of the 7 oz. mix is the same as T_2. This is clearly warmer than if we do not allow the milk to warm up; that is, we keep it in the fridge until we mix it which is Case 1. Hence $T_1 < T_2$.

Although solution 2 is short and elegant, it lacks the rigor of solution 1 (especially in the phrase "since it makes no difference if the fluids are mixed or not"). It is not our goal to show the correctness of either method (not least because it is difficult to justify Newton's law or its assumptions thermodynamically) but rather to show that the two solutions are *equally* correct!

THEOREM. *Not only does Newton's law imply that "it makes no difference if the fluids are mixed or not" but also, if there is* ***any*** *law of cooling for which it "makes no difference if the fluids are mixed or not", then it must be Newton's Law.*

We shall henceforth assume that there is *some* law of cooling. What we mean by that is that given an ambient temperature T_∞ and an object (e.g., a quantity of fluid), the future temperature of the object depends only on its present temperature and the elapsed time. Furthermore, we shall insist it is reasonable that temperature changes monotonically and continuously, converging at ∞ to T_∞. A mathematical way of saying all this is that there exists *some* monotonic and continuous function f with limit T_∞ at ∞ such that

$$\text{If } T(0) = f(t_0) \text{ then, for all } t,\ T(t) = f(t_0 + t). \tag{8}$$

Select any two temperatures $T_1 > T_2 > T_\infty$. Also let us identify the unique choices t_1 and t_2 for which $T_1 = f(t_1)$ and $T_2 = f(t_2)$. Finally, let $r = (T_2 - T_\infty)/(T_1 - T_\infty)$. Upon clearing the denominator, we observe that $(1 - r)T_\infty + rT_1 = T_2$. Let us envision mixing two quantities of fluid, an amount r of temperature T_1 with a quantity $1 - r$ at room temperature T_∞. If we mix first, we get $rT_1 + (1 - r)T_\infty = T_2 = f(t_2)$, and then allowing the fluid to cool for a time period t, gives $f(t_2 + t)$. But waiting first gives an amount r of $f(t_1 + t)$ to be mixed with an amount $1 - r$ of T_∞ yielding $rf(t_1 + t) + (1 - r)T_\infty$. If mixing does not matter, these must be equal so

$$rf(t_1 + t) + (1 - r)T_\infty = f(t_2 + t).$$

This can be rewritten:

$$f(t_2 + t) - T_\infty = r(f(t_1 + t) - T_\infty). \tag{9}$$

Subtract $T_2 - T_\infty = r(T_1 - T_\infty)$, and divide by t to get

$$\frac{f(t_2 + t) - T_2}{t} = r\frac{f(t_1 + t) - T_1}{t}.$$

Assuming that f is differentiable, we may take the limit as $t \to 0$ to find

$$f'(t_2) = rf'(t_1). \tag{10}$$

Now set $t = 0$ in (9) and divide equation (10) by these equal quantities to get

$$\frac{f'(t_2)}{f(t_2) - T_\infty} = \frac{f'(t_1)}{f(t_1) - T_\infty}.$$

Since this holds for any time $t_2 > t_1$, the fractions must have a constant value, say $-k$,

$$\frac{f'(t_2)}{f(t_2) - T_\infty} = -k.$$

Upon integrating and exponentiating, we have

$$\begin{aligned}\ln(f(t_1+t)-T_\infty) &= -kt+c\\ f(t_1+t) &= T_\infty + Ce^{-kt}\\ T(t) = f(t_1+t) &= T_\infty + (T_1-T_\infty)e^{-kt}.\end{aligned}$$

We have in fact derived Newton's Law of Cooling. However in this approach we assumed, quite reasonably, that f is differentiable. We can avoid this assumption and succeed using the weaker assumption that f is continuous by using the technique of *functional equations*, a topic seldom seen in the undergraduate curriculum. To use this approach, set $s = t_2 - t_1$ in equation (9) and divide by the same equation with t replaced by 0 to get

$$\frac{f(t_1+s+t)-T_\infty}{f(t_1+s)-T_\infty} = \frac{f(t_1+t)-T_\infty}{f(t_1)-T_\infty}.$$

Next multiply both sides by

$$\frac{f(t_1+s)-T_\infty}{f(t_1)-T_\infty}$$

to give us

$$\frac{f(t_1+s+t)-T_\infty}{f(t_1)-T_\infty} = \frac{f(t_1+s)-T_\infty}{f(t_1)-T_\infty}\cdot\frac{f(t_1+t)-T_\infty}{f(t_1)-T_\infty}.$$

This suggests that we can define a new function $g(t) := (f(t_1+t)-T_\infty)/(f(t_1)-T_\infty)$ for $t \geq 0$ to reveal the functional equation

$$g(s+t) = g(s)g(t), \qquad g(0) = 1. \tag{11}$$

This equation is (one of several) known as Cauchy's equation (see [**1**]) and, assuming only the continuity of $g(x)$ it turns out that there exists a real number k such that

$$g(t) = e^{-kt}. \tag{12}$$

This is remarkable, since infinite differentiability then follows from the much weaker condition of continuity (in fact, it follows from the even weaker condition of boundedness on an interval and continuity at a single point!—see [**1**] for details). We sketch a proof assuming continuity for all x. As a first consequence, we can choose $c \neq 0$ with $g(c) > 0$. By induction and equation (11),

$$g(nc) = g(c)^n.$$

Since $g(c) = g(n(c/n)) = g(c/n)^n$, taking the nth root of both sides,

$$g(c/n) = g(c)^{1/n}.$$

Then $g(mc/n) = g(c/n)^m = g(c)^{m/n}$ and so, for every positive rational x,

$$g(xc) = g(c)^x. \tag{13}$$

Since $g(-xc)g(xc) = g(-xc+xc) = g(0) = 1$, equation (13) holds for all rational numbers x. By the continuity of g, the two functions on either side of equation (13) are defined and continuous for each real number x and agree on the dense set of rational

numbers, and so equation (13) holds for all x. Since there is some k such that $g(c) = e^{-kc}$, we may rewrite (13) as equation (12).

To apply this to the previous problem, recall that $g(t) := (f(t_1 + t) - T_\infty)/(f(t_1) - T_\infty)$ satisfies equation (11). Using its solution (12), and the definition $T(t) := f(t_1 + t)$, we may write

$$T(t) = T_\infty + (T(0) - T_\infty)e^{-kt}$$

for some k—Newton's Law!

Going the other way, assume Newton's law (2). Then two similar fluids at temperatures a and b respectively satisfy temperature laws

$$\begin{cases} T_1(t) = T_\infty + (a - T_\infty)e^{-kt} \\ T_2(t) = T_\infty + (b - T_\infty)e^{-kt} \end{cases}$$

respectively. If mixed in the proportion $r : (1 - r)$, the resulting fluid has temperature law:

$$T(t) := T_\infty + (ra + (1 - r)b - T_\infty)e^{-kt} = rT_1(t) + (1 - r)T_2(t)$$

which shows that mixing does not matter. ■

Equation (2) can also reasonably be called "Newton's law of heating" when $T(0) < T_\infty$. The extension to that case follows from Principle A and that mixing does not matter: if a fluid at initial temperature $T_1(0) = T_\infty + c$ and another at temperature $T_2(0) = T_\infty - c$ are mixed in equal proportions, then the mixed temperature is constant T_∞. Since the first fluid obeys Newton's law ($T_1(t) = T_\infty + ce^{-kt}$) and since mixing does not matter, $T_2(t) = T_\infty - ce^{-kt}$. Since $c = T_\infty - T_2(0)$,

$$T_2(t) = T_\infty + (T_2(0) - T_\infty)e^{-kt}.$$

We have purposely not addressed many of the assumptions necessary for Newton's law to give even a reasonable approximation to reality. That which makes Newton's law of cooling interesting (to Calculus teachers at least) is its simplicity and, as we tried to show, its inevitability given a few basic principles and a little knowledge of the Calculus or of functional equations.

What then was Newton's intuition? In his paper of 1701, written in Latin, no equations appear. He wrote, however (this quoted from [**2**]): "the iron was laid not in a calm air, but in a wind that blew uniformly upon it . . . for thus equal parts of air were heated in equal times, and received a degree of heat proportional to the heat of the iron." The results of this experiment in 'forced convection' led to the empirical law equivalent to (1) and (2). We recommend the paper [**2**] and its references for further information.

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