Towards Generalizing Thrackles to Arbitrary Graphs

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Abstract

In the 1950s, John Conway came up with the notion of *thrackles*, graphs with embeddings in which no edge crosses itself, but every pair of distinct edges intersects each other exactly once. He conjectured that $|E(G)| \le |V(G)|$ for any thrackle G, a question unsolved to this day. Here, we discuss some of the known properties of thrackles and contribute a few new ones.
Only a few sparse graphs can be thrackles, and so it is of interest to find an analogous notion that applies to denser graphs as well. In this paper we introduce a generalized version of thrackles called *near-thrackles*, and prove some of their properties. We also discuss a large number of conjectures about them which seem very obvious but nonetheless are hard to prove. In the final section, we introduce *thrackleability*, a number between 0 and 1 that turns out to be an accurate measure of how far away a graph is from being a thrackle.

On Thrackles

Definition

*A **thrackle drawing** is a graph embedding where no edge crosses itself, but every pair of distinct edges intersects each other **exactly** once; this point of intersection is allowed to be a common endpoint, but cannot be tangential between the two edges. A **thrackle** is a graph that admits a thrackle drawing.*

If G is a thrackle, then any subgraph G' of G is also a thrackle.

Known Results

Theorem

If G is a linear thrackle (constrained to be drawn only using straight lines), then $|E(G)| \le |V(G)|$. (Pach et al, 2011)

The proof of this theorem is due to Pach and Sterling [?].

Theorem

If Conway's Conjecture is false, then a minimal counterexample will be topologically homeomorphic to one of the following three shapes, as shown in Wehner[?].

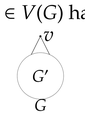
Furthermore, Rubinstein (unpublished) showed that if any one of these counterexamples exists, then so do the other two.

The Chromatic Number of A Thrackle

Theorem

A thrackle G has chromatic number at most 3.

We use induction on the number of vertices n. Clearly for $n \in \{1,2,3\}$, the result is trivial. Suppose now for $n \ge 4$ we have some thrackle G on n vertices. First we claim that G has a vertex of degree at most 2. If not, then all vertices in G have degree at least 3, and so summing the degrees we get $2|E(G)| = \sum_{v \in V(G)} \deg(v) \ge 3|V(G)|$, so that $|E(G)| \ge 1.5|V(G)|$, which contradicts the known bound of $|E(G)| \le 1.428|V(G)| < 1.5|V(G)|$ due to Fulek and Pach [?]. So assume $v \in V(G)$ has degree at most 2.



Consider the graph $G' = G \setminus \{v\}$. Clearly G' is a subgraph of G and is therefore a thrackle. By the induction hypothesis, G' is 3-colorable. Now when we add v back in to form G from G', we can extend a proper 3-coloring of G' to a proper 3-coloring of G, since the neighbors of v in G use up at most two of our three available colors. Hence G is 3-colorable as well, completing the proof.

On Near-Thrackles

Definition

*For any graph G, a **near-thrackle drawing** of G is an embedding of G in the plane satisfying the following conditions:*

- *First, out of all the embeddings of G, choose only the ones that maximize the number of pairs of edges that cross exactly once.*
- *Then, out of these embeddings of G, choose only the ones that maximize the number of pairs of edges that do not cross.*
- *Iterate the process by maximizing the number of pairs of edges that cross $2, 3, 4, \ldots$ times.*

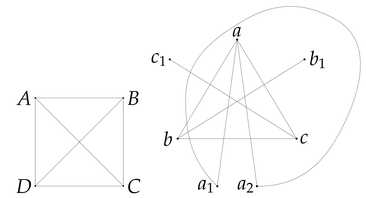


Figure : Examples of a near-thrackle drawing of a graph G

Conjecture (Weak deletion for edges)

Suppose we have a near-thrackle drawing of a graph G. Then there exists some $e \in E(G)$ such that deleting e from this drawing yields a near-thrackle drawing of $G \setminus \{e\}$.

Conjecture (Weak deletion for vertices)

Suppose we have a near-thrackle drawing of a graph G. Then there exists some $v \in V(G)$ such that deleting v from this drawing yields a near-thrackle drawing of $G \setminus \{v\}$.

Note that the conjecture for edges does *not* imply the one for vertices in case of weak deletion.

On Linear Near-Thrackles

Definition

*For any graph G, a **linear near-thrackle drawing** of G is a near-thrackle embedding of G subject to the constraint that all edges must be drawn as straight lines.*

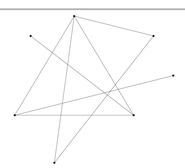


Figure : An example of a linear near-thrackle.

Theorem

A linear near-thrackle drawing of K_n is obtained by taking the n vertices in convex position, and then drawing all possible edges between them.

In general, this seems to be the unique linear near-thrackle drawing of K_n. A nice consequence of the theorem about complete graphs above is that linear near-thrackle drawings of K_n have $\binom{n}{4} + n\binom{n-1}{2} = \frac{n(n-1)(n-2)(n+9)}{24}$ pairs of edges that cross exactly once, and the remaining pairs do not cross at all.

Conjecture

A linear near-thrackle drawing of $K_{m,n}$ is obtained by taking $m+n$ vertices in convex position, and then defining m contiguous ones as one side of the partition, the n others as the other side of the partition, and drawing all possible edges between them.

If the conjecture is true, a similar expression can be obtained for $K_{m,n}$. It can be easily checked that in this case, linear near-thrackle drawings of $K_{m,n}$ have $m\binom{n}{2} + n\binom{n}{2} + \binom{n}{2}\binom{m}{2}$ pairs of edges that cross exactly once.

Conjecture (Weak deletion for edges, linear case)

Suppose we have a linear near-thrackle drawing of a graph G. Then there exists some $e \in E(G)$ such that deleting e from this drawing yields a linear near-thrackle drawing of $G \setminus \{e\}$.

Conjecture (Weak deletion for vertices, linear case)

Suppose we have a linear near-thrackle drawing of a graph G. Then there exists some $v \in V(G)$ such that deleting v from this drawing yields a linear near-thrackle drawing of $G \setminus \{v\}$.

It is worth observing that the conjectures for the general case do not, in fact, imply the ones for the linear case. We leave it to the reader to verify this straightforward fact.

Thrackleability

Definition

*The **thrackleability** $\varphi(G)$ of a graph G with at least two edges is defined as the quantity*

$$\frac{m_1}{\binom{|E(G)|}{2}},$$

where m_1 is the number of pairs of edges in $E(G)$ that cross each other exactly once in any near-thrackle drawing of G. Suppose $\mathscr{F} = (G_1, G_2, \cdots)$ is a family of graphs of increasing size, and the quantity $\varphi(G_n)$ converges to a finite value as $n \to \infty$. Then we denote this value by $\varphi(\mathscr{F})$.

It is clear that $\varphi(G)$ is bounded below by 0 and above by 1. Note that $\varphi(G) = 1$ if and only if G is a thrackle. Furthermore, note that $\varphi(G) \ne 0$ for any G, since any graph with two edges has a drawing in which $m_1 \ge 1$.

Proposition

If $\mathscr{F} = (G_1, G_2, \cdots)$ is the family of triangle-wedges, then $\varphi(\mathscr{F})$ exists and is bounded above by 8/9.

We know by our earlier arguments that G_2 is not a thrackle. So in every copy of G_2 in G_n for $n \ge 2$, there is a pair of edges that do not cross each other exactly once. In particular, note that for any two triangles in G_n, we have a unique copy of G_2, and hence a pair of edges corresponding only to that copy of G_2 that do not intersect exactly once. So, there are at least $\binom{n}{2}$ pairs of edges in G_n that do not intersect each other, and so

$$\varphi(G_n) \le 1 - \frac{\binom{n}{2}}{\binom{3n}{2}}.$$

In the limit $n \to \infty$, therefore, we get

$$\varphi(\mathscr{F}) \le 1 - \lim_{n\to\infty} \frac{\binom{n}{2}}{\binom{3n}{2}} = 1 - \frac{n^2}{9n^2} = \frac{8}{9}.$$

Definition

*The **linear thrackleability** $\varphi^+(G)$ to be $m_1'/\binom{|E(G)|}{2}$, where m_1' is the number of pairs of edges that intersect each other exactly once in a linear near-thrackle drawing of G.*

As before, $0 < \varphi^+(G) \le 1$ for any G, with $\varphi^+(G) = 1$ if and only if G is a linear thrackle, for instance, any odd cycle.

Proposition

Let $\mathscr{G}$ be the family of complete graphs $\{K_n\}_{n\in\mathbb{N}}$. Then, $\varphi^+(\mathscr{G}) = 1/3$, where φ^+ for a family of graphs is defined as for φ.

$$\varphi^+(\mathscr{G}) = \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24}}{\binom{\binom{n}{2}}{2}} = \frac{8}{24} = \frac{1}{3}.$$

This raises an obvious question. How close to 0 can $\varphi(G)$ or $\varphi^+(G)$ get? Are there specific families of graphs for which asymptotically φ or φ^+ gets arbitrarily close to 0? If not, what is a good lower bound for φ or φ^+? Can we get $\varphi(G)$ to be at most 1/2?

Acknowledgments and References

I would like to thank Rik Sengupta, an incredible mentor who helped me with the Thrackle Conjecture. Also, I want to credit Andrey Grinshpun who provided helpful suggestions to prove a theorem, and Dr. Douglass Woodall who kindly shared his research paper on thrackles with me. Furthermore, I would like to extend my gratitude to the PRIMES-USA program staff, namely Dr. Ben Elias, PRIMES-USA director, Dr. Tanya Khovanova, who provided me with great feedback, Dr. Slava Gerovitch, PRIMES director, and Dr. Jacob Fox, who originally inspired me to solve this problem.

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