Almost Pythagorean Triples

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Recently I was squaring and adding terms of the sequence $5, 10, 15, 20, 25, 30, \ldots$. I noted that $10^2 + 15^2 = 18^2 + 1$, $20^2 + 25^2 = 32^2 + 1$, $25^2 + 35^2 = 43^2 + 1$. I kept getting a square plus one. This suggested solving the Diophantine equation:

$$x^2 + y^2 = z^2 + 1. \tag{1}$$

Let's call a solution of (1) in integers an **Almost Pythagorean Triple (APT)**. In analytic geometry (1) represents a hyperboloid of revolution of one sheet, a doubly ruled surface. The tangent plane at a point cuts this surface in two straight lines called rulings. There are two sets of rulings, and each line of one set meets no other line of that set, but meets every line of the other set except one line, to which it is parallel.

The transformation which sends the point (x, y, z) into $(-x, -y, -z)$ sends each ruling into a parallel ruling of the other set. We shall show that the integral solutions of (1) lie on such rulings, and that we can use this fact to find all APT's. With a table of squares you can make a list of APT's. The first six are: (5,5,7), (4,7,8), (8,9,12), (7,11,13), (11,13,17), (10,15,18). You may notice that integers with last digit 2,3,7,8 repeatedly appear as the z-coordinate of an APT. Those ending in 2 or 7 form an arithmetic progression, as do the corresponding x and y coordinates. Likewise with those ending in 3 or 8. These arithmetic progressions give us the formulas:

$$x = 3t + 2, \qquad y = 4t + 1, \qquad z = 5t + 2 \tag{2}$$

$$x = 3t + 1, \qquad y = 4t + 3, \qquad z = 5t + 3. \tag{3}$$

We see that (1) is satisfied when we substitute these formulas in it. Integral values of t in (2) and (3) give us infinitely many APT's. Formulas (2) and (3) are sets of parametric equations of two parallel lines with direction numbers 3,4,5 (a primitive Pythagorean triple), lying in the surface (1).

Generalizing this, we show that if a, b, c is any primitive Pythagorean triple, that is, $\mathrm{GCD}(a, b, c) = 1$, then there are a pair of parallel rulings on (1) with direction numbers a, b, c, and that all APT's lie on such rulings.

First we find the direction numbers of the two rulings through any APT (p, q, r). If the parametric equations of a ruling through (p, q, r) are $x = At + p$, $y = Bt + q$, $z = Ct + r$, then $(At + p)^2 + (Bt + q)^2 = (Ct + r)^2 + 1$ for all t implies that:

$$A^2 + B^2 = C^2 \tag{4}$$

$$Ap + Bq = Cr. \tag{5}$$

Solving (4) and (5) for A, B, C in terms of p, q, r, we find the direction numbers A, B, C and A', B', C' of the two rulings.

$$\begin{aligned} A &= pr + q \qquad & A' &= pr - q \\ B &= qr - p \qquad & B' &= qr + p \\ C &= r^2 + 1 \qquad & C' &= r^2 + 1. \end{aligned} \tag{6}$$

These direction numbers are not necessarily primitive Pythagorean triples. We can get primitive triples from the formulas in (6) by dividing by the GCD. It is known that we can get all primitive Pythagorean triples from the formulas: $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$, where u and v are relatively prime, and one is odd and the other even.

Given a primitive Pythagorean triple a, b, c we show how to write the equations of the two rulings on (1) with direction numbers a, b, c as follows:

$$\begin{aligned} x &= at + p, & y &= bt + q & z &= ct + r \\ x &= at + p' & y &= bt + q' & z &= ct + r', \end{aligned} \tag{7}$$

where $p + p' = a$, $q + q' = b$, $r + r' = c$. One can check that if one set of formulas in (7) satisfies (1), so does the other. Since from (6) c divides $r^2 + 1$, we have $r^2 \equiv -1 \pmod{c}$. This congruence will have two solutions r and r' with $r + r' = c$, since each prime factor of c is of the form $4n + 1$. See [**3**]. With one of these values of r we get from (5)

$$ap + bq = cr. \tag{8}$$

Solving this linear Diophantine equation for p and q, and finding p' and q' from $p + p' = a$, $q + q' = b$, we may write the equations (7) of the two rulings. Since every APT lies on such a ruling we can find them all. One can take the same approach to solve in integers the more general equation:

$$x^2 + y^2 = z^2 + s^2, \tag{9}$$

where s is a fixed integer.

Some APT's are isosceles, like $(5,5,7)$. We can find all isosceles APT's by solving the Pell equation:

$$x^2 - 2y^2 = -1. \tag{10}$$

See [**3**]. Solutions of (10) are given by the even numbered convergents of the continued fraction for $\sqrt{2}$. The first three isosceles APT's are: $(5,5,7)$, $(29,29,41)$, $(169,169,239)$. The odd numbered convergents give solutions of the other Pell equation:

$$x^2 - 2y^2 = 1. \tag{11}$$

These will give isosceles **Nearly Pythagorean Triples (NPT)**, that is, solutions of the Diophantine equation:

$$x^2 + y^2 = z^2 - 1. \tag{12}$$

Some examples of NPT's are: $(2,2,3)$, $(4,8,9)$, $(12,12,17)$, $(8,32,33)$, $(10,50,51)$, $(22,46,51)$, $(34,38,51)$. Geometrically, equation (12) represents a hyperboloid of two sheets, which has no real rulings. The method used to find APT's does not work for NPT's. However, each APT determines an NPT, and conversely.

THEOREM. *If (p, q, r) is an APT, then $(2pr, 2qr, 2r^2 + 1)$ is an NPT, and if (p, q, r) is an NPT, then $(2p^2 + 1, 2pq, 2pr)$ is an APT.*

This is easily checked by substituting in equations (1) and (12). Applying this to the formulas for APT's given by equations (2) and (3) gives us the formulas for NPT's:

$$x = 30t^2 + 32t + 8, \qquad y = 40t^2 + 26t + 4, \qquad z = 50t^2 + 40t + 9 \tag{13}$$

$$x = 30t^2 + 28t + 6, \qquad y = 40t^2 + 54t + 18, \qquad z = 50t^2 + 60t + 19. \tag{14}$$

These formulas give us infinitely many NPT's. Similarly, the equations for any pair of parallel rulings on (1) give rise to formulas for NPT's. But not every NPT can be found in this way.

Equation (12) is the special case where $w = 1$ of the equation:

$$x^2 + y^2 + w^2 = z^2. \tag{15}$$

A complete solution of (15) was found by Catalan in 1885. See [**1**]. This solution is: $x = 2(pr + qs)$, $y = 2(qr - ps)$, $w = p^2 + q^2 - r^2 - s^2$, $z = p^2 + q^2 + r^2 + s^2$. If $w = 1$, then setting $r^2 + s^2 = n$, we get $p^2 + q^2 = n + 1$. In this case, a solution of (12), that is, an NPT, is given by $x = 2(pr + qs)$, $y = 2(qr - ps)$, $z = 2n + 1$. We can find an NPT whenever we have two consecutive integers n and $n + 1$, each of which is a square or the sum of two squares. A complete solution of (12) in rational numbers rather than integers is given by $(x/w, y/w, z/w)$, where x, y, z, w are given by Catalan's formulas.

References

[1] A. B. Ayoub, Integral solutions to the equation $x^2+y^2+z^2=u^2$: a geometric approach, this MAGAZINE, 57 (1984) 222-223.

[2] L. E. Dickson, History of the Theory of Numbers, 4th edition, 2, Chelsea, New York, 1966, p. 266.

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Simultaneous Triangle Inequalities

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It is a well-known result that the necessary and sufficient conditions that three positive numbers be the lengths of the sides of some triangle are that

$$b+c>a, \qquad c+a>b, \quad a+b>c. \tag{1}$$

Clearly it then follows that

$$(b+c-a)(c+a-b)(a+b-c)>0. \tag{2}$$

Also, it is easy to see that for $a, b, c>0$, (2) $\Rightarrow$ (1). For at most one of the three factors in (2) can be $\leqslant 0$ and this would violate (2). The latter inequality is also equivalent to

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)>0$$

or, by multiplying out, to

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)>0. \tag{3}$$

By Heron's formula for the area F of a triangle [1], (3) is given more compactly as

$$16F^2>0. \tag{3'}$$

As an extension of the above results, one can ask does there exist a polynomial inequality in the n positive numbers $a_1, a_2, \ldots, a_n$ which implies that any three of the numbers are lengths of sides of a triangle. Offhand one would expect that such a polynomial inequality exists and also that its degree is at least of order kn. Surprisingly, there is such a polynomial of degree 4 for all $n>3$. Formally, our result is as follows:

If $a_1, a_2, \ldots, a_n>0$ for $n \geqslant 3$ and

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)^2>(n-1)\left(a_1^4+a_2^4+\cdots+a_n^4\right), \tag{4}$$

then a_i, a_j, a_k, for all $i \neq j \neq k$, are lengths of sides of a triangle.

Our proof is by induction. First we show that (4) implies that

$$\left(a_2^2+a_3^2+\cdots+a_n^2\right)^2>(n-2)\left(a_2^4+a_3^4+\cdots+a_n^4\right), \tag{5}$$

where the left out term, a_1, is arbitrary. After some elementary algebra involved in completing a square, (4) can be shown to be equivalent to

$$0>\left\{a_1^2-S_2/(n-2)\right\}^2-\left\{S_2^2-(n-2)S_4\right\}\{n-1\}/\{n-2\}^2,$$

where $S_m=a_2^m+a_3^m+\cdots+a_n^m$. Consequently, (4) $\Rightarrow$ (5). Then by induction,