An Invariant for Singular Links

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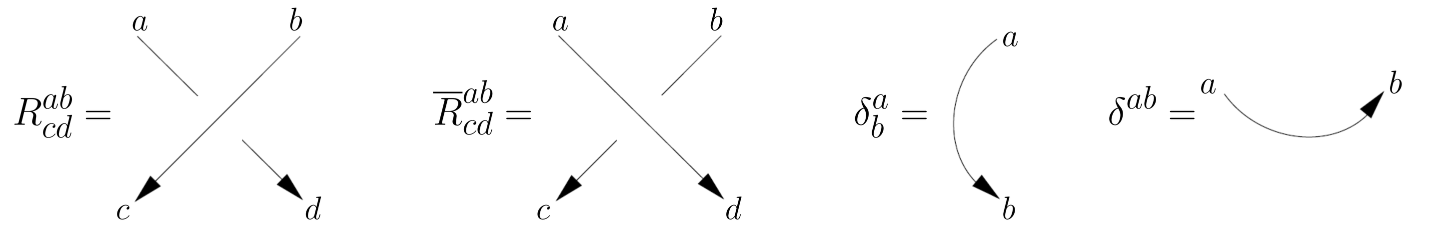
1. Background

Sections of oriented knots can be represented as diagrammatic tensors in the following manner:



where $a, b, c, d \in I$. These along with analogous representations for different orientations can give a tensor representation for any knot diagram by evaluating each crossing as a tensor.

2. States and the Invariant $T(K)$

Definition 1. *A state of a knot, K, is a mapping $\sigma : E(K) \longrightarrow I$ where E is the edge set of a diagram of K.*

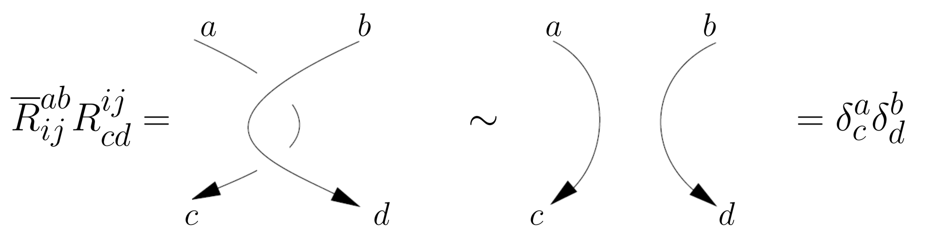
Definition 2. *We define*

$$T(K) = \sum_{\sigma} \langle K|\sigma\rangle .$$

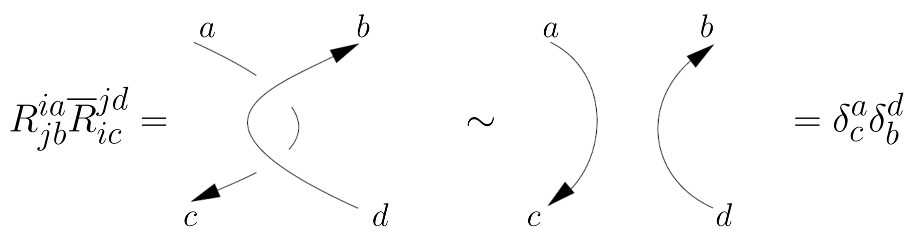
Here σ runs over all the states of K, and $\langle K|\sigma\rangle$ denotes the product of the vertex weights R^{ab}_{cd}, $\overline{R}^{ab}_{cd}$ assigned to the crossings in K in the given state.

We wish for $T(K)$ to be a invariant under the Reidemeister moves and thus R^{ab}_{cd}, $\overline{R}^{ab}_{cd}$ when summed over the index set I must satisfy:

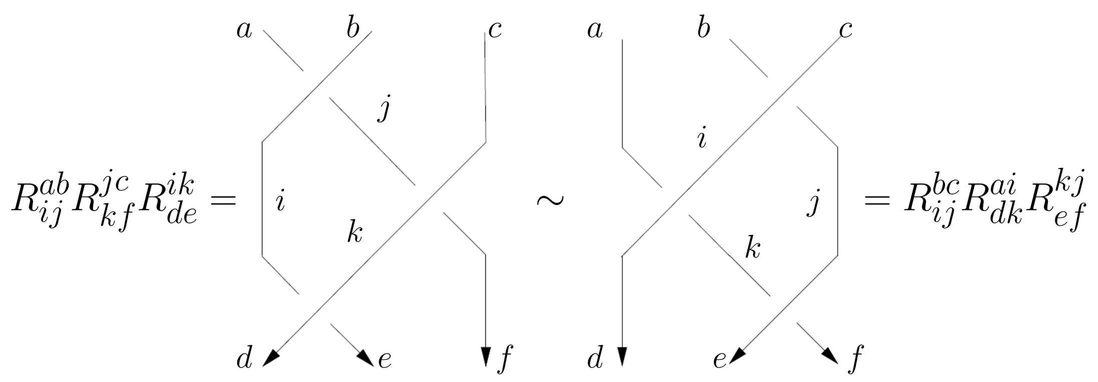
Channel Unitarity-



Cross-Channel Unitarity-



Given channel and cross-channel unitarity it suffices to impose only the Reidemeister III condition with all positive crossings-



This last relation gives the Yang-Baxter Equation:

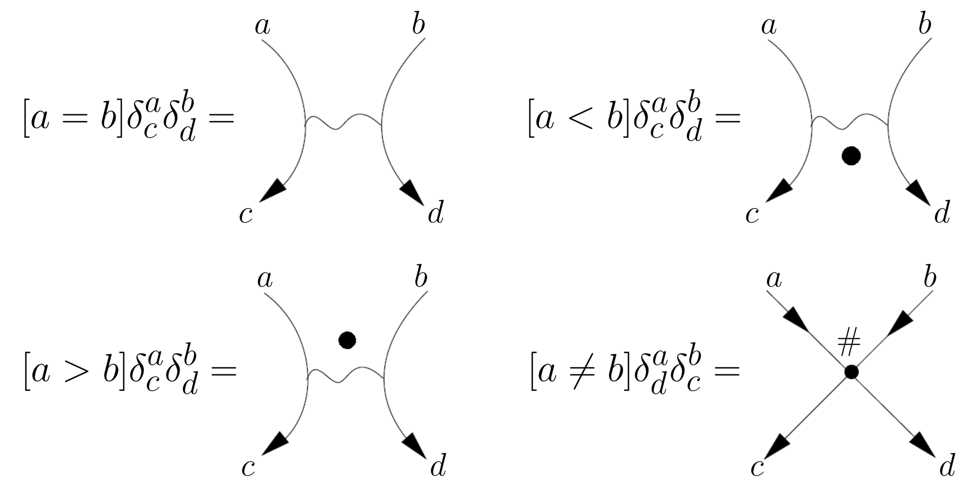
$$\sum_{i,j,k \in I} R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = \sum_{i,j,k \in I} R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}.$$

A similar relation is imposed on $\overline{R}$.

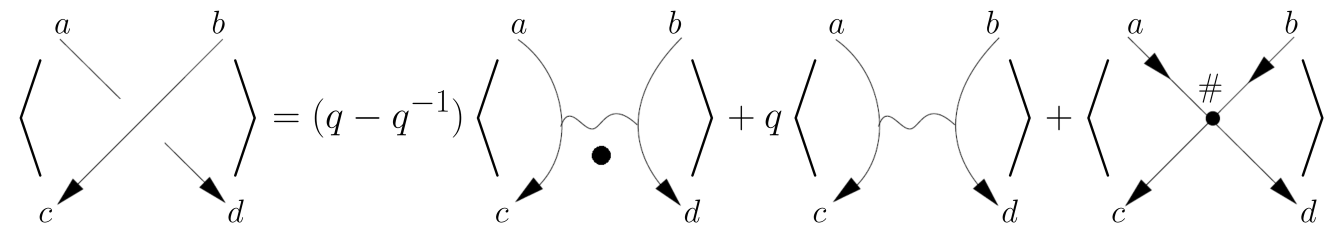
Theorem 1. *If the matrices R and $\overline{R}$ satisfy channel and cross-chanel unitarity and the Yang-Baxter equation, then T(K) is a regular isotopy invariant for oriented knot diagrams.*

3. A Solution to the Yang-Baxter Equation

We define the following diagrams as



Here $a, b, c,$ and d belong to the index set $I = \{1-n, 3-n, \ldots, n-3, n-1\}$.
Using q as an indeterminate, we define the relation at a single crossing in a knot as



Notationally this becomes

$$R^{ab}_{cd} = (q - q^{-1})[a < b]\delta^a_c \delta^b_d + q[a = b]\delta^a_c \delta^b_d + [a \neq b]\delta^a_d \delta^b_c.$$

Similarly we define

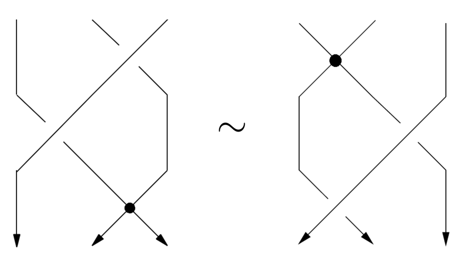
$$\overline{R}^{ab}_{cd} = (q^{-1} - q)[a > b]\delta^a_c \delta^b_d + q^{-1}[a = b]\delta^a_c \delta^b_d + [a \neq b]\delta^a_d \delta^b_c.$$

Proposition 1. *R^{ab}_{cd} and $\overline{R}^{ab}_{cd}$ defined in this way satisfy channel and cross-channel unitarity and the Yang-Baxter equation. Thus R and $\overline{R}$ can be used to construct a Yang-Baxter type regular isotopy invariant for singular links and knotted graphs.*

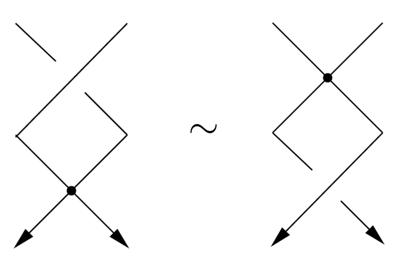
4. Extending the Solution to Singular Knots

Definition 3. *A singular link is an immersion of a disjoint union of circles in $\mathbb{R}^3$ which admits only finitely many singularities that are all transverse double points.*

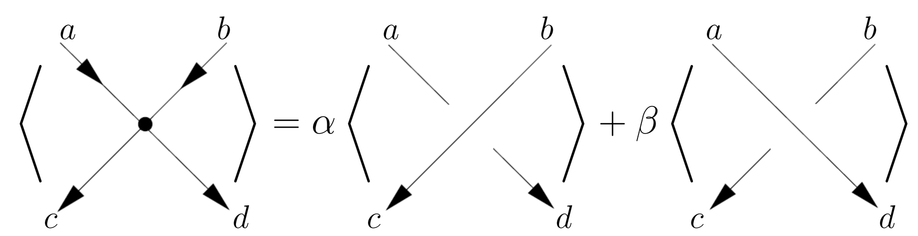
To begin our research we wanted our evaluation, $\langle K\rangle$, to be an invariant for singular links. It must be invariant under additional Reidemeister moves involving flat crossings:
-Reidemeister 4:



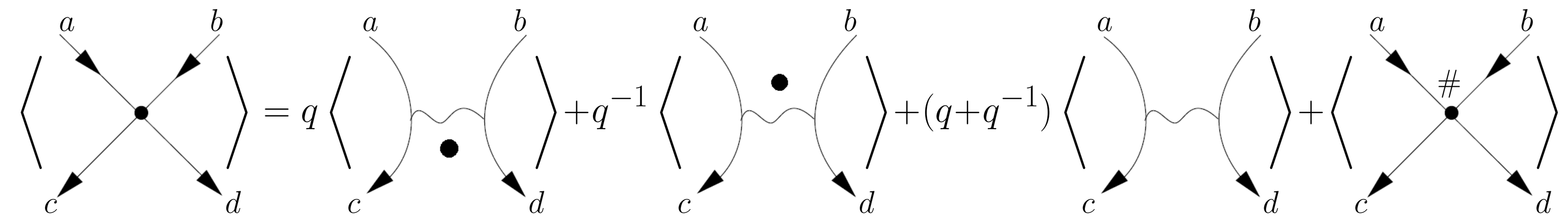
-Reidemeister 6:



In attempt to resolve vertices in such a way that these two equations are satisfied we see that,



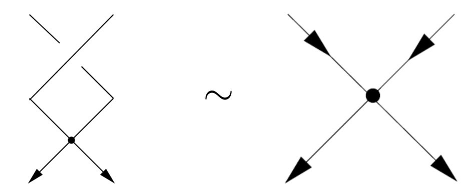
is invariant under Reidemeister 4 and Reidemeister 6 moves for any α and β. By choosing $\alpha = \frac{q}{q-q^{-1}}$ and $\beta = \frac{-q^{-1}}{q-q^{-1}}$ we obtain:



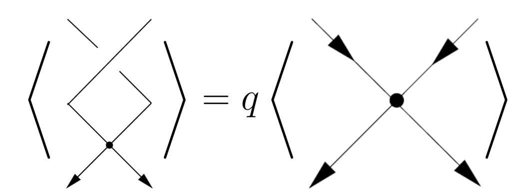
and thus a regular isotopy polynomial invariant $< G >$ for singular links G.

5. Reidemeister 5

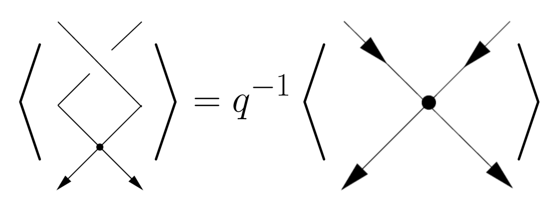
In order to have an invariant for cross-like oriented topological 4-valent knotted graphs we must have an evaluation of flat crossings such that the following is satisfied.
-Reidemeister 5:



Using the resolution obtained in section 4 we obtain,



Similarly,



By defining the writhe, $\omega(K)$, to be the number of positive crossings, R, minus the number of negative crossings, $\overline{R}$, we can define an invariance under this Reidemeister 5 move.

Theorem 2. *Let $P(K) = q^{-\omega(K)}\langle K\rangle$. Then $P(K)$ is an ambient isotopy invariant for cross-like oriented topological 4-valent knotted graphs.*

6. Additional Results and Future Research

- In the previous section we only considered cross-like oriented vertices, but it is possible to generalize $\langle K\rangle$ to include alternating oriented vertices. Is it possible to extend $P(K)$ to topological knotted graphs?
- The polynomial $< G >$ satisfies similar graphical relations involving planar graphs as those for the $sl(n)$ polynomial for knots. What relations can be drawn between the two constructions?
- Finally, is there another unique construction of such an invariant?

7. Acknowledgments

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