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Summary This note provides an elementary solution of the brachistochrone problem. This problem is to find the curve connecting two given points so that an object slides without friction along the curve from one point to the other point in the least possible time. The key is to introduce a coordinate system where the expected cycloid solutions are built in.

A Property Characterizing the Catenary

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Let $y(x)$ be any strictly positive C^1 function, and consider the curve which is the graph of $y(x)$ over an interval $[a, b]$ in the function's domain. This curve has a well-defined arc length and there is a well-defined area under it. Are there any functions which have the property that the ratio of the area under the curve to the curve's arc length is independent of the interval over which they are measured?

In order for this property to hold, we must have [**1**, page 279]

$$\int_a^b y(x)\,dx = k\int_a^b \sqrt{1+y'(x)^2}\,dx,$$

where k is a positive constant independent of a and b. In order for this to be true for all intervals $[a, b]$ in the function's domain, the integrands must be identically equal. Bringing k inside the right-hand integral, setting the integrands equal, and solving for $y'(x)$ yields

$$y'(x) = \pm\frac{\sqrt{y(x)^2-k^2}}{k}. \tag{1}$$

Clearly $y(x) = k$ is a solution. When $y(x) \neq k$, we can separate variables and find a more surprising result:

$$y(x) = k\cosh\left(\frac{x-c}{k}\right),$$

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which is the well-known catenary curve. Therefore catenaries and constant functions are the only curves that are twice-differentiable everywhere and have the property that they bound an area proportional to their arc length over any horizontal interval. (If we relax our smoothness requirements, we could also allow curves that are defined piecewise to be either constant or catenary curves over different intervals).

We have obtained the geometric result that at every point on a catenary $y\,dx = k\,ds$, where ds is the arc length differential. This property leads directly to the interesting result that for every interval $[a, b]$, the geometric centroid of the area under a catenary curve defined on this interval is the midpoint of the perpendicular segment connecting the centroid of the curve itself and the x-axis. Note that the centroid of the curve lies above the curve itself.

The result that the content of a region bounded by a catenary is proportional to the content of the boundary itself over any interval extends directly to the three-dimensional case. If a surface of revolution has the property that the ratio of the volume it encloses to its surface area is independent of the interval on which it is defined, then it must obey the equation [**1**, pp. 326 and 466]

$$\int_a^b \pi y(x)^2\,dx = k\int_a^b 2\pi y(x)\sqrt{1+y'(x)^2}\,dx.$$

After setting the integrands equal as before, a factor of $\pi y(x)$ cancels from both sides and we can rearrange to obtain equation (1) again, but with each k replaced by the term $2k$. Therefore the surface of revolution generated by

$$y(x) = 2k\cosh\left(\frac{x-c}{2k}\right),$$

which is the famous catenoid surface, is the only twice-differentiable surface of revolution other than the cylinder of radius $2k$ which encloses a volume that is k times its surface area over every horizontal interval.

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Summary We show that the area under a catenary curve is proportional to its length in the following sense: given a catenary curve, we can take any horizontal interval and examine the ratio of the area under the curve to the length of the curve on that interval, and we find that the resulting ratio is independent of the chosen interval. This property extends to the three-dimensional case as well: the volume contained by a horizontal interval of a catenoid surface is proportional to its surface area in the same sense. We also show from this property that the centroid of the area under an interval of a catenary is the midpoint of the segment connecting the centroid of the catenary and the x-axis.