References

1. P. Blanchard, R. L. Devany, and G. R. Hall, *Differential Equations*, 3rd ed., Thompson, Brook/Cole, Belmont CA, 2006.
2. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 8th ed., John Wiley & Sons, Hoboken NJ, 2005.
3. C. H. Edwards and D. E. Penney, *Differential Equations & Linear Algebra*. Prentice Hall, Upper Saddle River NJ, 2001.
4. Z. H. Nitecki and M. M. Guterman, *Differential Equations with Linear Algebra*. Saunders College Publishing, Philadelphia, 1986.
5. R. E. Williamson, *Introduction to Differential Equations and Dynamical Systems*, 2nd ed., McGraw Hill, Boston, 2001.

Averaging Sums of Powers of Integers

Thomas J. Pfaff (tpfaff@ithaca.edu), Department of Mathematics, Ithaca College, Ithaca NY

In "An Anything-But-Average Average" [**1**], Kahan investigated the equation

$$\frac{\sum_{i=1}^{n} i^a + \sum_{i=1}^{n} i^b}{2} = \left(\sum_{i=1}^{n} i\right)^c. \tag{1}$$

He found three solutions $(a, b, c) = (1, 1, 1), (3, 3, 2), (5, 7, 4)$, for $a, b, c \leq 100$. We show here that these are the *only* solutions to (1), and also provide some necessary conditions for

$$\frac{\sum_{i=1}^{n} i^{a_1} + \sum_{i=1}^{n} i^{a_2} + \cdots + \sum_{i=1}^{n} i^{a_{m-1}}}{m-1} = \left(\sum_{i=1}^{n} i\right)^{a_m}, \tag{2}$$

to hold, which was left open in [**1**].

Averaging two sums of powers of integers We need two useful facts about

$$S_k(n) = \sum_{i=1}^{n} i^k,$$

which can be found in [**2**]: (i) The function $S_k(n)$ is a polynomial in n of degree $k+1$, and (ii) the leading coefficient of $S_k(n)$ is $1/(k+1)$. Note also that

$$\left(\sum_{i=1}^{n} i\right)^c = \left(\frac{n^2+n}{2}\right)^c,$$

and hence the leading term in (1) is $\frac{n^{2c}}{2^c}$.

Without loss of generality we assume that $a \leq b$. By (i) $b + 1 = 2c$, so that both sides of (1) have an n^{2c} term. If $a = b$, then by (ii) the leading coefficient of the left hand side of (1) is

$$\frac{\frac{1}{a+1} + \frac{1}{b+1}}{2} = \frac{1}{b+1} = \frac{1}{2c}.$$

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Now, as we noted, the leading coefficient of the right hand side of (1) is $1/2^c$. Hence, any solution must have $2c = 2^c$, so either $c = 1$ or $c = 2$. This leads to the solutions $(a, b, c) = (1, 1, 1), (3, 3, 2)$.

If $a < b$, then the leading coefficient of the left hand side of (1) is

$$\frac{\frac{1}{b+1}}{2} = \frac{1}{2(b+1)} = \frac{1}{4c}.$$

This time we must have $4c = 2^c$, which implies that $c = 4$. This yields $(a, b, c) = (5, 7, 4)$. Therefore, $(a, b, c) = (1, 1, 1), (3, 3, 2), (5, 7, 4)$ are the only solutions to (1).

Averaging more than two sums of powers of integers In order to deal with (2), we assume that $a_1 \le a_2 \le \cdots \le a_{m-1-p} < a_{m-p} = \cdots = a_{m-1}$. In other words, the last p of the a_i are equal and larger than the preceding a_i. By the same reasoning as above, we must have $a_{m-1} + 1 = 2a_m$, and the coefficient of the largest power on the left hand side of (2) is

$$\frac{\frac{p}{a_{m-1}+1}}{m-1} = \frac{p}{(m-1)2a_m}.$$

Setting this equal to the leading coefficient on the right hand side, and we get that any solution to (2) has

$$2^{a_m-1} = \frac{(m-1)a_m}{p}. \tag{3}$$

We can make some observations, on this basis, about possible solutions to (2). First, if $p = m - 1$, then (3) reduces to $2^{a_m-1} = a_m$. We then get that $a_m = 1$ or $a_m = 2$, which leads to trivial extensions to the previous solutions, that is, $a_1 = \cdots = a_m = 1$, and $a_1 = \cdots = a_{m-1} = 3$ with $a_m = 2$. So assume that $p < m - 1$. Now 2^{a_m-1} is exponential in a_m, while $\frac{(m-1)a_m}{p}$ is linear in a_m. Since $1 < (m-1)/p$, these two functions of a_m intersect only once, not necessarily at integer values of a_m. Therefore, there can be at most one solution to (2) for each m and p. If m is even, then equation (3) implies that $p = m - 1$; thus equation (2) has only the trivial solutions discussed above.

On the other hand, if $m = 2^x + 1$ and $p = 2^{x-1}$ we get $a_m = 4$. The case when $x = 1$ gave the solution $(5, 7, 4)$. In fact, Mathematica finds 92 solutions for (3) when a_m and m are between 1 and 100, and 956 solutions when $1 \le a_m \le 100$ and $1 \le m \le 1000$. Some of the first few solutions to (3), sorted by m, are $(m, a_m, p) = (3, 4, 1), (5, 3, 3), (5, 4, 2), (7, 4, 3), (9, 3, 6), (9, 4, 4)$. Notice that the first solution has $m = 3$ so it reduces to solving (1), but this only tells us that $c = 4$ and not how to find a and b. This is, of course, the $(5, 7, 4)$ solution above.

Concluding remarks As m increases, and the difference between m and p increases, the difficulty finding solutions also increases. We do have one possible solution in $(5, 3, 3)$ that we could check since we already know that $m = 5$, $a_m = 3$, and $a_2 = a_3 = a_4 = 5$, which leaves us with only one unknown term in a_1. When m and p differ by only 2 we will have only a_1 unknown. We also know that a_1 cannot exceed 3. It turns out that we do get a solution with $a_1 = 3$. At this point, it is not known if any of these solutions to (3) yield solutions to (2), beyond what is known here. One could write a program to search for solutions, but that won't solve the problem in general, and so it seems that solving this problem for $m > 3$ will require some other approach.

Summary. When is the average of sums of powers of integers itself a sum of the first n integers raised to a power? We provide all solutions when averaging two sums, and provide some conditions regarding when larger averages may have solutions.

References

1. S. Kahan, An Anything-But-Average Average, *Journal of Recreational Math.* **31** (2002–2003) 253–256.
2. G. Mackiw, A Combinatorial Approach to Sums of Integer Powers, *Math. Mag.* **73** (2000) 44–46; available at http://dx.doi.org/10.2307/2691490

Uncountably Generated Ideals of Functions

B. Sury (sury@isibang.ac.in), Stat-Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore, India

Undergraduates usually think that the study of continuous functions and the study of abstract algebra are divorced from each other. More often than not, they find it very surprising that concepts like rings and ideals could be applied to function spaces as well! Some applications in algebra texts concern the ring $C[0, 1]$ of real-valued continuous functions on $[0, 1]$; however, these texts restrict themselves to a few standard exercises although more could be accomplished with almost the same amount of labor. For instance, the exercises in [**1**, p. 388], [**2**, p. 259], and [**3**, p. 140] ask for a proof that maximal ideals in $C[0, 1]$ are not finitely generated. The fact that these maximal ideals are not *countably* generated does not seem to be as well-known as it should be although the proof is not harder! We will prove this, and then use it to produce some non-prime ideals in $C(0, 1)$ which cannot be countably generated as well. Without further ado, let us begin.

Maximal ideals in $C[0, 1]$ Let $I_c = \{f \in C[0, 1] : f(c) = 0\}$ where $c \in [0, 1]$. Contrary to what we want to prove, assume that I_c is generated by a countable set $\{f_1, f_2, \ldots\}$. By re-scaling, we may assume that $|f_n(x)| \leq 1$ for all x and for all n. Consider the function

$$f(x) := \sum_{n=1}^{\infty} \sqrt{\frac{|f_n(x)|}{2^n}}.$$

By uniform convergence, f is continuous. Clearly, $f \in I_c$. By assumption $f = \sum_{i=1}^{r} g_i f_i$ for suitable g_i in $C[0, 1]$ and natural number r.

Let M be an upper bound for $|g_i|$ for all $i \leq r$ and all x in $[0, 1]$. Then,

$$|f(x)| \leq M \sum_{i=1}^{r} |f_i(x)|.$$

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