A Markov Chain Analysis of the Game of Jai Alai

PHILIP J. BYRNE
College of St. Benedict
St. Joseph, MN 56374

ROBERT HESSE
University of Minnesota
Minneapolis, MN 55455

Introduction Jai Alai, a game resembling racquetball, evolved in Spain during the seventeenth century. In twentieth-century America, Connecticut, Florida, and Rhode Island operate Jai Alai frontons, where fans can watch the action and bet on the outcomes of games. While most fans find the play itself exciting, the real mathematical interest is the manner in which a winner is determined.

Before play begins, the eight players (or two-player teams) are placed in a queue with assigned post positions 1 to 8. A game consists of a sequence of short matches between two players; the first match pits player 1 against player 2. The winner of a match faces the next player in the queue, while the loser of a match returns to the back of the queue. The first seven matches are worth one point each; succeeding matches are worth two points. The winner is the first player to reach or exceed seven points.

Experienced Jai Alai bettors realize, intuitively, that players in low-numbered post positions have an advantage over players near the back of the queue. Informal analysis supports this point: for example, player 1 could win the game by winning the first seven matches, for one point each. Even if he loses an early match, player 1 will likely have a second opportunity to play. Player 6, on the other hand, could win by surviving his first five matches (the last three are worth two points each). But if player 6 loses any of these matches, another player may well reach seven points before player 6 returns to the front of the queue.

We will show how a Jai Alai game can be modelled as a Markov chain, and thus show how each player's winning depends on his post position. We will assume for convenience that all eight players have equal skill, but other assumptions about relative skills can be readily incorporated into the Markov chain analysis. The same approach can also be applied to other kinds of bets, such as trifectas and quinielas.

The model To model Jai Alai using Markov chains, we must first define an appropriate notion of a state. To describe the game at any time requires two data for each player: (1) his current position in the queue; and (2) his current score. Thus we assign to the ith position in the queue an ordered pair (a_i, b_i), with a_i the number of the player in position i, and b_i this player's current score. In particular, a_1 and a_2 are the players who will meet in the next match. We now define a state to be an ordered set $\{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5), (a_6, b_6), (a_7, b_7), (a_8, b_8)\}$. The initial state, for instance, is always $\{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0), (8,0)\}$. The second state must then be either $\{(1,1), (3,0), (4,0), (5,0), (6,0), (7,0), (8,0), (2,0)\}$ or $\{(2,1), (3,0), (4,0), (5,0), (6,0), (7,0), (8,0), (1,0)\}$.

We now assume that the outcomes of different matches are probabilistically independent of one another, and that the probability of player m winning a match against player n remains constant throughout the game. Then the probability of the game moving from one state to another depends only on the states, not on the previous history of the game. With these assumptions, a Jai Alai game becomes a Markov chain.

We'll need some basic notation and terminology and a fundamental result. First we order the states in some convenient manner, and assign them labels $1, 2, 3, \dots$. If the game is in state i after r matches, the transition probability of moving to state j on the next match is denoted by p_{ij}. For most states in our Jai Alai game there are only two other states to which they can move. For every state i of this type, $p_{ij} = 0$ for all but two j's, so the ith row of the transition matrix $P = (p_{ij})$ has only two nonzero entries. The only other possible states are those in which some player has amassed seven or more points, winning the game. For all such states i we will take $p_{ii} = 1$ and $p_{ij} = 0$ if $i \neq j$. These states are called absorbing.

For any Markov chain with a finite number of states, we can label the absorbing states with integers $1, 2, \dots, s$, and the nonabsorbing with integers $s+1$, $s+2, \dots, s+t$. Then the transition matrix P has the form

$$P = \begin{pmatrix} I_s & 0 \\ R & Q \end{pmatrix}$$

where I_s is the $s \times s$ identity matrix, 0 is an $s \times t$ matrix of zeros, and R and Q are $t \times s$ and $t \times t$ matrices, respectively. In particular, R gives transition probabilities from nonabsorbing to absorbing states, and Q gives transition probabilities from nonabsorbing to nonabsorbing states. It's a general fact that the (i, j)-entry of the $t \times s$ matrix $(I_t - Q)^{-1}R$ gives the probability that the Markov chain ends up in absorbing state j given that the initial state was $s + i$ (see the Appendix for the sketch of a proof). For our Jai Alai application, the initial state is always that in which the queue has the players in numerical order, and each player has zero points.

A three-player example To illustrate these ideas, consider first a simplified Jai Alai game, with only three players and two points needed for a win. The first two matches are worth one point each; a third match, if needed, is worth two points. By analogy with the eight-player game, a state is an ordered set of three ordered pairs $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$, where a_i and b_i denote the player number and current score for the player in the ith position in the queue. There are, in all, eleven possible states, which we label as follows:

Label	State
7	$\{(1,0), (2,0), (3,0)\}$
8	$\{(1,1), (3,0), (2,0)\}$
1	$\{(1,2), (2,0), (3,0)\}$
9	$\{(3,1), (2,0), (1,1)\}$
2	$\{(3,3), (1,1), (2,0)\}$
3	$\{(2,2), (1,1), (3,1)\}$
10	$\{(2,1), (3,0), (1,0)\}$
4	$\{(2,2), (1,0), (3,0)\}$
11	$\{(3,1), (1,0), (2,1)\}$
5	$\{(3,3), (2,1), (1,0)\}$
6	$\{(1,2), (2,1), (3,1)\}$

If the three players have equal ability, then the associated transition matrix for this three-player game is

state	1	2	3	4	5	state 6	7	8	9	10	11
1	1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0
7	0	0	0	0	0	0	0	.5	0	.5	0
8	.5	0	0	0	0	0	0	0	.5	0	0
9	0	.5	.5	0	0	0	0	0	0	0	0
10	0	0	0	.5	0	0	0	0	0	0	.5
11	0	0	0	0	.5	.5	0	0	0	0	0

From this we can compute directly the 5×6 matrix $(I_5 - Q)^{-1}R$. Its first row, (.25, .125, .125, .25, .125, .125), displays the probabilities of the Jai Alai game ending in the absorbing states 1, 2, 3, 4, 5, and 6, respectively, given the initial state 7. Since player 1 is the winner in absorbing states 1 and 6, player 2 in states 3 and 4, and player 3 in states 2 and 5, their respective probabilities of winning are .375, .375, and .25.

The eight-player game Analyzing the eight-player game is similar, but the transition matrix P is much larger. To find the number of states, we wrote two computer programs to count all the vertices in an appropriate tree diagram. The root vertex of this tree corresponds to the initial state of the game; two branches connect the root with two vertices representing the two states that can occur next. Since each match in a Jai Alai game has two possible outcomes, most vertices have two branches emanating from them. Any vertex corresponding to a winning state has no outgoing branches. Our first program counted 844,767 vertices in this tree diagram. In the eight-player game (unlike the three-player game) some states can be reached by more than one sequence of match outcomes. Our second program eliminated these duplications. We found, in the end, a total of 134,215 distinct states in the eight-player game.

Since we wish to compute $(I_t - Q)^{-1}R$, the size of the transition matrix P might seem to create computational problems. However, two useful observations come to our rescue. First, the matrix P is sparse: no row contains more than two nonzero entries. Sparse matrices often admit special, efficient algorithms for such operations as multiplication and inversion (see, e.g., **[1]**). Second, since the Jai Alai game has only one possible initial state, which we number $s + 1$, we need only compute the first row of $(I_t - Q)^{-1}R$ to determine the players' winning probabilities.

This can be done by finding the first row of $(I_t - Q)^{-1}$, which we denote by $(x_1, x_2, \ldots, x_t)$, and then multiplying by the matrix R. We can find $(x_1, x_2, \ldots, x_t)$ by comparing the first rows on both sides of the equation $(I_t - Q)^{-1}(I_t - Q) = I_t$, which gives us $(x_1, x_2, \ldots, x_t)(I_t - Q) = (1, 0, 0, \ldots, 0)$. Taking transposes yields the linear system $Ax = b$, where $A = (I_t - Q)^t$, $x = (x_1, x_2, \ldots, x_t)^t$, and $b = (1, 0, 0, \ldots, 0)^t$. The nonabsorbing states can be labelled in such a way that $I_t - Q$ is nearly upper triangular (in the three player game, $I_5 - Q$ *was* upper triangular). In this case, A is nearly lower triangular. Thus, after relatively few row operations, back-substitution can be performed, starting with x_1, to successively find values for $x_1, x_2, x_3, \ldots, x_t$.

The winning probabilities given below, which assume the players to be of equal ability, agree with those found by Moser [3] in her computer search through all possible games.

Player	Probability of Winning
1	.1631
2	.1631
3	.1386
4	.1240
5	.1020
6	.1026
7	.0888
8	.1177

The table has several interesting features. First, since players 1 and 2 begin the game at the front of the queue and play each other in the first match, symmetry of their situations naturally results in equal probabilities of winning. Note also that player 8 has a higher winning probability than players 5, 6, and 7. This reflects the fact that only player 8 can win the game by winning as few as four matches on his first turn to play. This more than compensates for player 8's smaller probability of getting a second chance to play after a loss. The table also supports the general intuition of Jai Alai bettors that players in low-numbered post positions have an advantage. Note, however, the small advantage of player 6 over player 5. A possible explanation is that player 5 needs a string of six wins, while player 6 would need to win only five matches.

Many related problems could be studied with the approach presented here. For example, Moser [3] used her computer search through the Jai Alai game tree to determine the probabilities for place, show, and exacta bets when all players have equal abilities, and also the probabilities of each player winning under certain combinations of unequally-skilled players. All of these situations could be handled using Markov chains. For place, show, and exacta bets, we would need to expand the number of possible states to account for the way ties are broken for place and show in Jai Alai. If the players have unequal skill, then for each ordered pair of players, (m, n), we would assign a probability, θ_{mn}, of player m winning a match against player n. The implication for the transition matrix P is straightforward. If the transition from state i to state j involves player m winning a match against player n, then $p_{ij} = \theta_{mn}$. This change from the earlier case affects only the nonzero entries of P, so P will again be a sparse matrix and the Markov chain analysis will remain computationally feasible.

Appendix To find the probability that a Markov chain ends up in a certain absorbing state given that it started in a particular nonabsorbing state, we note first that the (i, j) entry of the matrix R gives the probability of moving from nonabsorbing state $s+i$ to absorbing state j in one step. The (i, j) entry of the matrix QR gives the probability of moving from nonabsorbing state $s+i$ to some other nonabsorbing state in one step, and then to absorbing state j on the next step. That is, the entries of QR are the probabilities of moving from nonabsorbing states to absorbing states in two steps. Similarly, the entries of Q^2R give the probabilities of moving from nonabsorbing states to absorbing states in three steps, and so on. Therefore, the probability that the Markov chain eventually ends up in absorbing state j given that the initial state was $s+i$ is determined by the (i, j) entry of the matrix

$$R + QR + Q^2R + Q^3R + \dots = (I_t + Q + Q^2 + Q^3 + \dots)R.$$

It can be shown that all the entries of Q^n approach zero as n tends to infinity (see [**2**, pp. 43–45]). This condition yields the following matrix generalization of the familiar formula for the sum of a geometric series:

$$(I_t - Q)^{-1} = I + Q + Q^2 + Q^3 + \dots$$

(see [**2**, p. 22]). Thus we see that the (i, j)-entry of the $t \times s$ matrix $(I_t - Q)^{-1}R$ gives the probability that the Markov chain ends up in absorbing state j given that the initial state was $s + i$.

REFERENCES

1. G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd edition, Johns Hopkins University Press, Baltimore, 1989.
2. J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, Springer-Verlag, New York, 1976.
3. L. E. Moser, A mathematical analysis of the game of Jai Alai, *Amer. Math. Monthly* 89 (1982), pp. 292–300.

Poker With Wild Cards—A Paradox?

STEVE GADBOIS
Rhodes College
Memphis, TN 38112

I participate in a sporadic poker game whose organizer detests any use of wild cards. (A *wild card* can be called anything its holder wishes.) I'd always attributed this aversion to some personality quirk. Then I discovered a reason to share his concern.

After a recent class in which I tossed out an unsubstantiated claim about wild cards sometimes altering the accepted hierarchy of poker hands, I decided I'd better actually do the calculations before my students did. I wasn't surprised to substantiate my claim, but I was surprised to discover that unresolvable inconsistencies can arise when wild cards are used. This note shows how, in one common situation, *no matter what hierarchy is established, the resulting probabilities are incompatible with it*. So perhaps my friend (who happens to be a political scientist, as well as the frequent victor in our always-friendly games) has more innate mathematical talent than either of us realized.

The usual hierarchy of poker hands (when played without wild cards) is, from best to worst, royal flush, straight flush, four-of-a-kind, full house, flush, straight, three-of-a-kind, two pair, one pair, and junk.[1] Without wild cards, this hierarchy is consistent

[1]Some of these terms may not be self-explanatory. A *royal flush* consists of an ace, king, queen, jack, and ten, all in one suit. A *straight flush* comprises five in a row, all in one suit (but not a royal flush). A *full house* includes three-of-a-kind and one pair. A *flush* consists of five cards in one suit (but not a royal flush or a straight flush). A *straight* has five in a row (but not a royal flush or a straight flush). Any other hand is *junk*.