A Generalization of Fibonacci Far-Difference Representations

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SMALL 2013, Williams College

Previous Results

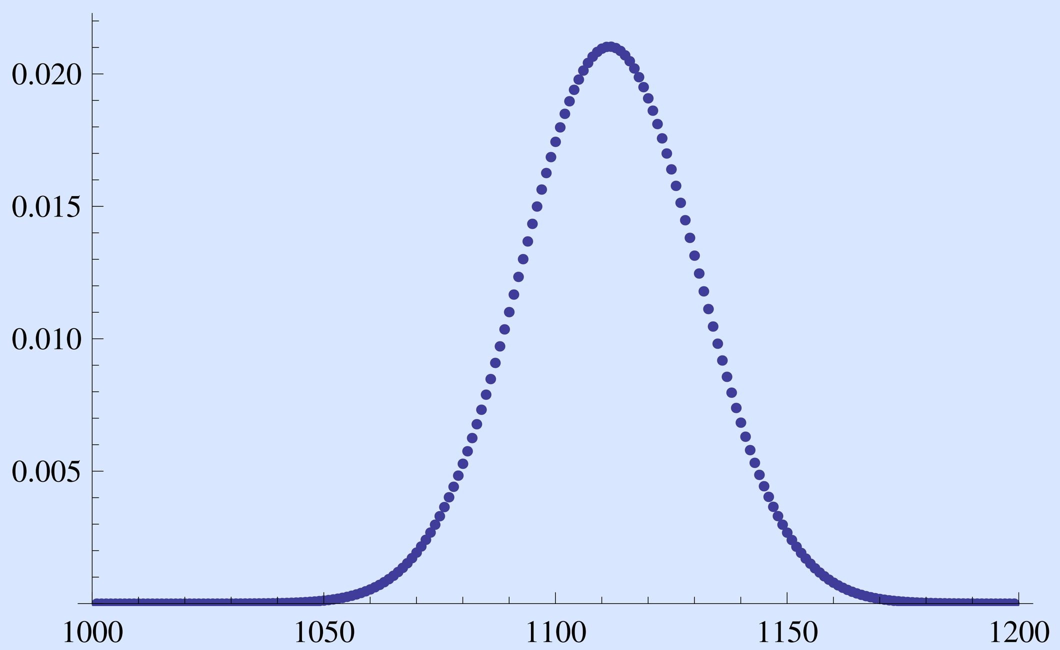
Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$; $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.



Number of summands in $[F_{2010}, F_{2011})$

Far-Difference Representations

A sequence $\{a_n\}$ is said to have an (s, d) far-difference representation if every integer can be written uniquely as sum of terms $\pm a_n$ in which every two terms of the same sign are at least s apart in index and every two terms of opposite sign are at least d apart in index.

Acknowledgements

This research was conducted as part of the 2013 SMALL REU program at Williams College and was partially supported by NSF grant DMS0850577 and Williams College. We would like to thank our adviser, Professor Steven J. Miller, for his guidance and our colleagues from PANTHER 2013.

The k-Skipponaccis

The k-Skipponaccis are recurrence relations of the form

$$S_{n+1} = S_n + S_{n-k}$$

for some $k \geq 0$ and initial terms $1, 2, 3, \ldots, k-1, k$.

Theorem 1

Every $x \in \mathbb{Z}$ has a unique (s, d) far-difference representation for the k-Skipponnacis such that $s = 2k + 2$ and $d = k + 2$.

Gaussianity Results

We are interested in the distribution of $\mathcal{K}_n$ and $\mathcal{L}_n$, the numbers of positive and negative summands for all integers in a single interval $(R_n, R_{n+1}]$, where

$$R_n = S_n + S_{n-(2k+2)} + S_{n-2(2k+2)} + \ldots$$

Our approach is to use generating functions. Let $p_{n,m,\ell}$ denote the number of numbers in the interval $(R_n, R_{n+1}]$ whose decomposition consists of m positive summands and ℓ negative summands. Then the generating function is given by:

$$\begin{aligned} G(x, y, z) &= \sum p_{n,m,\ell} x^m y^\ell z^n. \\ &= \frac{xz - xz^2 + xyz^{k+3} - xyz^{2k+3}}{1 - 2z + z^2 - (x+y)\left(z^{2k+2} + z^{2k+3}\right) - xy\left(z^{2k+4} + z^{4k+4}\right)} \end{aligned}$$

Theorem 2

For each non-negative pair $(a, b) \neq (0, 0)$, $X_n = a\mathcal{K}_n + b\mathcal{L}_n$ converges to a Gaussian as $n \to \infty$. Moreover, the mean and variance of X_n both grow linearly with n.

Corollaries: As $n \to \infty$,

- $\mathcal{K}_n, \mathcal{L}_n$ converge to normal distributions.
- The expected values of $\mathcal{K}_n$ and $\mathcal{L}_n$ both grow linearly with n and differ by a constant, as do their variances.
- $\mathcal{K}_n + \mathcal{L}_n$ and $\mathcal{K}_n - \mathcal{L}_n$ are independent.

Finding the Gaps

We also study the distribution of index gaps between summands. Let $P_n(j)$ be the probability that the size of a gap between adjacent terms in the far-difference decomposition of a number $m \in (R_{n-1}, R_n]$ is of length j.

Theorem 3

As $n \to \infty$, the probability $P_n(j)$ converges to geometric decay for $j \geq 2k + 2$, with computable limiting values for other j.

Sequences with (s,d) Decompositions

For fixed positive integers s, d, define a sequence $\{a_n\}_{n=1}^{\infty}$ by

i. For $n = 1, 2, \ldots, \min(s, d)$, let $a_n = n$.

ii. For $\min(s, d) < n \leq \max(s, d)$, let

$$a_n = \begin{cases} a_{n-1} + a_{n-s} & \text{if } s < d \\ a_{n-1} + a_{n-d} + 1 & \text{if } d \leq s. \end{cases}$$

iii. For $n > \max(s, d)$, let $a_n = a_{n-1} + a_{n-s} + a_{n-d}$.

Theorem 4

The sequence $\{a_n\}$ defined above has an unique (s, d) far-difference representation for the given choice of s and d.

Corollaries:

- The Fibonacci sequence has a $(4, 3)$ far-difference representation by writing

$$F_n = F_{n-1} + F_{n-2} = F_{n-1} + (F_{n-3} + F_{n-4})$$

- The k-Skipponacci has a $(2k + 2, k + 2)$ far-difference representation by writing

$$S_n = S_{n-1} + S_{n-k-1} = S_{n-1} + (S_{n-k-2} + S_{n-2k-2}).$$

- Every integer can be represented uniquely as a sum of powers of threes, since the recurrence relation

$$a_n = 3a_{n-1} = (a_{n-1} + a_{n-1} + a_{n-1})$$

defines the sequence $(1, 3, 9, 27, \ldots)$ with $s = d = 1$.