NOTES

Power Distribution in Four-Player Weighted Voting Systems

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The Hometown Muckraker is a small newspaper with a few writers and layout personnel, and an editorial staff of four. When major policy decisions require that the editorial staff vote, the Editor-in-Chief gets 3 votes; the Managing Editor gets 2 votes; and the News Editor and Feature Editor each get 1 vote, for a total of 7. A majority (4 votes) is needed to pass a motion.

Lately, however, the Managing Editor has begun to feel that she has no more say in policy decisions than the News Editor or the Feature Editor has. In a private meeting with the Editor-in-Chief, she was told, "That's ridiculous; you have twice as many votes as they do, and therefore you have twice as much say." Is the Editor-in-Chief right? Or is it in fact true that the News Editor and the Feature Editor have just as much influence as the Managing Editor?

In this note we show that not only is the Editor-in-Chief wrong, but there is no way to remedy the problem, at least in the sense of restructuring the voting system so that there is the following hierarchy of influence, or *say* on policy matters:

(i) The Editor-in-Chief has the most influence, but less than half of the *total influence*.

(ii) The Managing Editor has less say than the Editor-in-Chief but more influence than the other two editors.

(iii) The News and Feature Editors have equal influence, but less than their senior editors.

Now consider a second scenario: Pennsylvania has decided to secede from the United States and invites its neighbors to do the same. There are three takers: Ohio, New York, and West Virginia. The four states decide to keep the existing geographical boundaries in place and form the Republic of PAIN (Pennsylvania and its Neighbors).

An electoral college system is instituted to elect the head of the executive branch of government, who is called the Big Cheese. An integer number of electoral votes for each state is chosen which is roughly proportional to population. (How to fairly apportion the total number of votes among the four states is another problem entirely; potential solutions to that problem have been much studied, and we shall assume that the framers of the Republic of PAIN are oblivious to the complexity of that problem and have apportioned the electoral votes in a manner satisfactory to them. Balinski and Young **[1]** provide a comprehensive treatment of the apportionment problem.) A candidate must receive a strict majority of the electoral votes in order to become the

Big Cheese. These votes are distributed as follows:

New York	38
Pennsylvania	25
Ohio	23
West Virginia	4

Since there are a total of 90 electoral votes, 46 are required for a victory.

A provision is in place to reapportion the electoral votes based on population changes. However, the population of each state continues to grow at a (universally) constant percentage rate, so that the four states maintain their ratios to the total population. However, after a few Big Cheese elections, West Virginia legislators propose a restructuring of the electoral college system, because they have begun to believe that their citizens actually have no influence whatsoever in Big Cheese elections. Unfortunately, their claim falls on deaf ears, because the other three states argue that the small number of electoral votes West Virginia casts is perfectly appropriate for its size and that these four votes *do* matter. Who is right?

Perhaps in this second scenario it is easier to see the validity of the objection: Any two of Ohio, New York, or Pennsylvania can agree on a Big Cheese candidate, and that candidate will win. Conversely, no candidate can win the election without carrying two of these states. As we will argue more carefully in the next section, it is in fact true that West Virginia's votes *don't* count; the key point here is that West Virginia can never cast the *deciding* votes. But if it is inappropriate for the Republic of PAIN to allocate votes in proportion to population, then how should the voting system be restructured to give West Virginia some measure of influence befitting its size? Or will it turn out, as in the previous example, that the desired objective cannot be achieved?

The Banzhaf Power Index The tool which we shall use to measure influence in each of our two hypothetical examples is called the *Banzhaf power index*, developed by attorney John Banzhaf [**2**] in the 1960s to argue that among the six districts of Nassau County, New York, represented by the Board of Supervisors, only the three largest districts wielded any real influence. (See also chapter 2 of the book by Tannenbaum and Arnold [**4**].) Informally speaking, to measure a voting party's influence, we ask, "How likely is it (indeed, does it occur at all) that this party can cast deciding votes?"

Following Tannenbaum and Arnold [**4**], we assume a finite number of voting parties called *players*, denoted $P_1, P_2, \ldots, P_n$. Player P_i casts a positive integer number of votes, v_i. The number of votes q needed to pass a motion shall be called the *quota*, and we assume

$$\frac{v_1 + \cdots + v_n}{2} < q \leq v_1 + \cdots + v_n.$$

We shall call this a *weighted voting system* (WVS) of size n, and represent it by

$$[q; v_1, v_2, \ldots, v_n]$$

and assume $v_1 \geq v_2 \geq \cdots \geq v_n$.

We shall also stipulate that $v_i < q$ holds for each i; otherwise it would be possible for some P_i to vote alone and pass a motion. But in addition, we assume that for all i,

$$\sum_{j \neq i} v_j \geq q.$$

If this were not true for some i, then P_i would have the power to prevent any motion from passing; the votes of all the other players would not exceed the quota. We call such a condition *veto power*.

A *coalition* in a WVS is a subset of the players. A *winning coalition* is a coalition for which the combined votes of the players exceed the quota. Otherwise, we have a *losing coalition*. We call a player in a winning coalition *critical* if the removal of that player results in a losing coalition. The presence of a critical player in a winning coalition will be called a *critical instance*; if a winning coalition has k critical players, we shall say that there are k critical instances corresponding to that coalition.

We are now ready to define the *Banzhaf power index* for a player P_i as the ratio of the number of instances in which P_i is critical to the total number of critical instances. We shall denote this ratio by $B(P_i)$. Note that since each $B(P_i)$ is a percentage, we have

$$B(P_1) + \cdots + B(P_n) = 1.$$

The aim of the Banzhaf power index is to measure the percentage of the total amount of power each player in the WVS possesses. The underlying philosophy is that power is characterized by the ability to cast deciding votes, and if there is a player who can never do this, then that player has no effective power.

There are other measures of power in weighted voting systems, such as the Shapley-Shubik power index, which in fact predated the Banzhaf index by a decade. The Shapley-Shubik model, however, is better suited to situations for which the order in which the players vote is a factor. In that case, one considers all $n!$ permutations of the n-player coalition; associated to each one is a unique player whose votes bring the cumulative total up to (or past) the quota, if all players vote in favor of a motion. That player is considered critical. The Shapley-Shubik power index for P_i is then the total number of instances in which P_i is critical, divided by $n!$.

The Banzhaf and Shapley-Shubik power distributions for a given WVS can sometimes agree, but they can also be dramatically different. (Chapter 9 of Taylor's book [**5**] provides an example, and also other models of power.) We have chosen the Banzhaf model here because we have no reason to distinguish the various permutations of a given winning coalition. Our tacit assumption is that all players vote at once, and each possesses no knowledge of how the others are voting.

With the aid of Banzhaf's power index, let us verify that West Virginia is powerless in the Republic of PAIN example, represented by the WVS $[46; 38, 25, 23, 4]$. The players here are P_1, P_2, P_3, and P_4, corresponding to New York, Pennsylvania, Ohio, and West Virginia, respectively. The winning coalitions, along with their *weights* (total numbers of votes cast), are as follows:

Coalition	Weight
$\{P_1, P_2\}$	63
$\{P_1, P_3\}$	61
$\{P_2, P_3\}$	48
$\{P_1, P_2, P_3\}$	71
$\{P_1, P_2, P_4\}$	67
$\{P_1, P_3, P_4\}$	65
$\{P_2, P_3, P_4\}$	52
$\{P_1, P_2, P_3, P_4\}$	90

Observe that no winning coalition to which West Virginia (P_4) belongs has a weight exceeding the quota by less than 4 votes. For this reason, West Virginia is not a critical player in any winning coalition, so that $B(P_4) = 0$. This is already an interesting finding, but let us look closer, and note that in each 2-player winning coalition, each player is critical. This gives 6 critical instances so far. Then when we examine $\{P_1, P_2, P_3\}$, we see that the removal of any one player from this coalition results in a 2-player winning coalition; even without arithmetic we can see this by referring to the list of 2-player winning coalitions. Hence $\{P_1, P_2, P_3\}$ yields no critical instances. For the other 3-player winning coalitions, however, we find

- P_1 and P_2 are critical in $\{P_1, P_2, P_4\}$
- P_1 and P_3 are critical in $\{P_1, P_3, P_4\}$
- P_2 and P_3 are critical in $\{P_2, P_3, P_4\}$

This gives another 6 critical instances. The 4-player winning coalition has no critical players (so that veto power is absent in this WVS), and we have a total of 12 critical instances, with each of P_1, P_2, and P_3 critical in 4 instances. Hence we find $B(P_1) = B(P_2) = B(P_3) = 1/3$.

The surprising revelation behind our analysis, then, is that not only does West Virginia have no power, but the other three states have equal power. So despite the proportional representation apparently afforded by the electoral college system, the allocation of votes may as well be as follows: 0 to West Virginia and 1 to each of the other states, with 2 electoral votes needed to become the Big Cheese. (According to our stipulation that each v_i should be positive, we would not consider $[2; 1, 1, 1, 0]$ to be an acceptable WVS. However, any WVS of the form $[2m; m, m, m, 1]$, with $m \geq 2$, would yield the same distribution of power as we find in the Republic of PAIN example.) The outcomes of all elections would be the same.

No wonder, then, that West Virginia proposes a change. The good news is that change is possible; the bad news, as the following result reveals, is that the options are quite limited.

THEOREM. *In any 4-player WVS with no veto power, there are only five possible power distributions:*

(a) $B(P_i) = 1/4$ *for every* i.

(b) $B(P_4) = 0$ *and* $B(P_i) = 1/3$ *for* $i \neq 4$.

(c) $B(P_1) = 1/2$ *and* $B(P_i) = 1/6$ *for* $i \neq 1$.

(d) $B(P_1) = B(P_2) = 1/3$ *and* $B(P_3) = B(P_4) = 1/6$.

(e) $B(P_1) = 5/12$, $B(P_2) = B(P_3) = 1/4$, *and* $B(P_4) = 1/12$.

Moreover, if $v_3 = v_4$, *then alternatives (b) and (e) are not possible.*

The reader can check that alternative (c) holds in the Hometown Muckraker example, so that indeed, the Managing Editor has no more power in the Banzhaf model than the News and Feature Editors have. Furthermore, it is impossible for the editorial staff to effect a hierarchy of power that is consistent with rank.

In the Republic of PAIN example, alternative (b) holds, and it would seem that the only sensible way to restructure the WVS so that all four states have some power, in amounts commensurate with their populations, is to effect alternative (e). The WVS $[11; 7, 5, 4, 2]$ does this, and if Ohio should object to having only twice as many electoral votes as West Virginia, even though its population is over six times larger, then perhaps the WVS $[13; 8, 6, 6, 1]$ would be acceptable; the power distribution still corresponds to alternative (e). Another example is $[90; 80, 65, 20, 5]$, in which Pennsylvania casts over three times as many votes as Ohio, a condition which might be psychologically disturbing to Ohio residents. Yet the power index for the two states is the same. The point is that, given two players P_i and P_j, the ratio v_i/v_j is not a reliable indicator of comparative power.

To further reinforce the point, consider the 3-player WVS $[13; 12, 11, 2]$. All 2-player coalitions are winning coalitions, and in all three cases, both players are critical. But in the 3-player coalition, none of the players is critical. Thus the total number of critical instances is 6, and each player is critical in two instances. Hence each player has power index $1/3$. This is true despite the fact that P_1 casts six times as many votes as P_3.

We include a fairly uninspiring proof of the theorem; one objective of this note is to stimulate interest in finding a slicker way to analyze weighted voting systems of a given size. We have treated size 4 because the brute force argument we give is not too cumbersome. One pleasing aspect of the proof is the revelation that one need only consider the various combinations of winning coalitions, and the proof relies only on the absence of veto power and not on the actual value of the quota q. Nor must one ever consider by how many votes a coalition wins; one need only know whether the coalition wins or loses. The proof also reveals that no matter the size of a WVS, only finitely many power distributions are possible (a fact which may be readily evident to a combinatorialist), so that in general, it is not possible to construct a WVS with a prescribed power distribution.

Proof of the Theorem We show that only five power distributions are possible in a 4-player WVS by exhausting the possible compositions of the set of winning coalitions. We use three important consequences of the absence of veto power:

1. The 4-player coalition is a winning coalition that does not yield any critical instances (since if one player were critical, then that player would have veto power). Therefore all critical instances occur among the 2- and 3-player winning coalitions.
2. All of the 3-player coalitions must be winners; if not, the missing player has veto power, since the 4-player coalition is certainly a winner. Hence the various cases are distinguished completely by which 2-player coalitions are winners. (We must point out, however, that changing the collection of 2-player winning coalitions will affect instances of criticality in 3-player winning coalitions, so the 3-player coalitions must still be examined.)
3. In any 2-player winning coalition, both players must be critical, since we do not permit single-player winning coalitions.

Our work is further simplified by observing that critical instances occurring in 3-player winning coalitions can always be detected by referring to the list of winning 2-player coalitions.

Since we need only consider cases distinguished by which 2-player coalitions win, let us start by observing the following rankings, in descending order of total weight, for 2-player winning coalitions.

- $\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_4\}, \{P_3, P_4\}$
- $\{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}$

Note that what is uncertain is how $\{P_1, P_4\}$ and $\{P_2, P_3\}$ compare. However, if one of these coalitions wins, then the other loses. In fact, if any 2-player coalition wins, then its complement loses; for example, if $\{P_1, P_2\}$ wins, then $\{P_3, P_4\}$ must lose, for otherwise we would have

$$v_1 + v_2 \geq q \quad \text{and} \quad v_3 + v_4 \geq q,$$

so that the strict majority requirement

$$q > \frac{v_1 + v_2 + v_3 + v_4}{2}$$

cannot hold for the quota.

It follows that there cannot be more than three 2-player winning coalitions in a 4-player WVS and that the coalitions $\{P_2, P_4\}$ and $\{P_3, P_4\}$ always lose.

We now consider the various cases. First, if there are no 2-player winning coalitions, then every player is critical in each of the four 3-player winning coalitions. Hence the total number of critical instances is 12, and each player is critical three times, so that $B(P_i) = 1/4$ for every i, which is alternative (a).

Next suppose that there is only one 2-player winning coalition; then it must be $\{P_1, P_2\}$, so that P_1 and P_2 are critical in this coalition and also in the coalitions $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$. On the other hand, P_3 and P_4 are not critical in these last two coalitions but are critical in the other two 3-player coalitions. Also, P_1 is critical in $\{P_1, P_3, P_4\}$, and P_2 is critical in $\{P_2, P_3, P_4\}$. Hence we have 4 critical instances for each of P_1 and P_2 and 2 critical instances for each of P_3 and P_4, so that alternative (d) results.

If there are two winning 2-player coalitions, then they must be $\{P_1, P_2\}$ and $\{P_1, P_3\}$, and we note that this can only occur if $v_3 > v_4$ (for if $v_3 = v_4$, then $\{P_1, P_3\}$ winning would imply that $\{P_1, P_4\}$ wins as well). P_1 is critical in each of the three 3-player coalitions containing P_1, yielding 5 critical instances for P_1. Then P_2 is critical in $\{P_1, P_2, P_4\}$ and $\{P_2, P_3, P_4\}$; P_3 is critical in $\{P_1, P_3, P_4\}$ and $\{P_2, P_3, P_4\}$; and the only 3-player coalition for which P_4 is critical is $\{P_2, P_3, P_4\}$. Again we have 12 critical instances in total, and we find that alternative (e) holds in this case.

There are now two cases involving three winning 2-player coalitions. We may have $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_4\}$ winning, yielding 6 critical instances so far. Then we observe that only P_1 is critical in the coalitions $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, and $\{P_1, P_3, P_4\}$, while all three players are critical in $\{P_2, P_3, P_4\}$. The result is alternative (c).

With $v_3 > v_4$ we may now encounter the case in which $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_2, P_3\}$ win. Note that here, P_4 is never a critical player, because removing P_4 from any 3-player coalition to which it belongs leaves one of the three winning 2-player coalitions. Hence $B(P_4) = 0$. Furthermore, removing any player from $\{P_1, P_2, P_3\}$ still leaves a winning coalition, but in the other three 3-player coalitions containing P_4, the other two players are critical. Here, as in every case, we have a total of 12 critical instances, with each of P_1, P_2, and P_3 critical 4 times. Hence we find $B(P_1) = B(P_2) = B(P_3) = 1/3$, so that alternative (b) holds. The proof is complete. ■

Next steps We pose the following questions for investigation:

- For weighted voting systems of size n, is there a formula in terms of n for the number of feasible power distributions? (In case $n = 3$, a simple check reveals that all three players must have equal power, so that only one power distribution is possible in the absence of veto power.)
- With a complete enumeration of the power distributions feasible for weighted voting systems of size n, can one efficiently generate a complete list of feasible power distributions for size $n + 1$ weighted voting systems?
- If a certain power distribution is desired, can one efficiently construct a WVS with the feasible power distribution that comes closest to the ideal (by some measure)? We might expect that the solution to this problem is dependent on the choice of norm used to measure deviation from the ideal power distribution. Such norm-dependence

does arise in the apportionment problem mentioned above. One incarnation of that problem is to fairly mete out congressional representatives to states in direct proportion to the state's population. The ideal share for each state, however, is in general not an integer, so given a positive integer partition of the total number of representatives, one wishes to measure the overall deviation from the ideal. An article by Ernst [**3**] gives a thorough discussion.

- If a player is to be removed from an n-player WVS, can an $(n-1)$-player WVS be efficiently constructed so as to preserve, as nearly as possible, the existing distribution of power among the remaining players?

One final observation is that in the case of a 4-player WVS, a *strict hierarchy* of power is impossible; that is, there are always at least two players with the same Banzhaf power index. The least value of n for which we can have

$$B(P_1) > B(P_2) > \cdots > B(P_n)$$

turns out to be $n = 5$. An example, due to the author, Christopher Carter Gay, and Jabari Harris, is the WVS [9; 5, 4, 3, 2, 1], which yields Banzhaf power distribution

$$\left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25}\right).$$

(The reader can check that the WVS [8; 5, 4, 3, 2, 1] does not induce the above Banzhaf power distribution.) We have also found, in unpublished work, that this is the *only* power distribution, of the 35 possibilities in the 5-player case, with strict hierarchy of power. What we have not yet found is an efficient way to analyze the 5-player problem in order to establish the number of power distributions feasible or generate the complete list of these power distributions.

The most naive attempt to effect strict hierarchy of power with $n = 6$, the WVS [11; 6, 5, 4, 3, 2, 1], fails to deliver, for this WVS yields power distribution

$$\left(\frac{9}{28}, \frac{1}{4}, \frac{5}{28}, \frac{3}{28}, \frac{3}{28}, \frac{1}{28}\right).$$

However, [15; 9, 7, 4, 3, 2, 1] yields power distribution

$$\left(\frac{5}{12}, \frac{3}{16}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \frac{1}{24}\right).$$

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