Generating Nonpowers by Formula

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For x a real number the expression $[x]$ as usual denotes the greatest integer $t \leqslant x$. We note that $[x+0.5]$ is the integer closest to x except when x is equidistant from two integers. The symbol N denotes the set of all positive integers.

For each integer $n>1$ we define the function f_n: $N \to N$ by $f_n(i) =$ the ith positive integer which is not a perfect nth power. For example in the table below we see the behavior of f_2; namely, f_2 lists in ascending order the elements in N which are not perfect squares.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...
$f_2(i)$	2	3	5	6	7	8	10	11	12	13	14	15	17	18	19	20	21	22	23	24	26	...

On pages 97 and 98 of [**1**], an explicit formula

$$f_2(i) = i + [\sqrt{i} + 0.5] \quad \text{for every } i \in N \tag{1}$$

is derived for f_2, and on pages 98 and 99 a quite different formula

$$f_2(i) = i + \left[\sqrt{i + [\sqrt{i}\,]}\,\right] \quad \text{for every } i \in N \tag{2}$$

is also established. The formula (1) is mentioned in [**2**]. In the present paper, we establish for each integer $n>1$ an explicit formula for f_n.

Whereas we have no interest in the sequence $\langle f_n(1), f_n(2), f_n(3), \dots \rangle$ *per se*, inasmuch as this sequence can be obtained by deleting the perfect nth powers from the sequence $\langle 1,2,3,\dots \rangle$, the idea of a simple formula capable of generating these sequences $\langle f_n(i) \rangle$ is an intriguing one. Indeed, we find it remarkable that a single simple formula using only the operations of addition, subtraction, multiplication, division, powers, roots, and the greatest integer function can skip over exactly those integers necessary to realize such a sequence.

LEMMA. *Let $n>1$ be an integer. Then*

$$f_n(k^n) = k^n + k. \tag{3}$$

Proof. Among the integers in the set $Q = \{1,2,\dots,k^n+k\}$, there are exactly k perfect nth powers: $1^n, 2^n, \dots, k^n$. Clearly, there are exactly $(k^n+k)-k = k^n$ non-nth-powers in Q, and thus $f_n(k^n) = k^n + k$.

THEOREM. *For each pair $n>1$ and $i>0$ of integers, let $p(n,i)$ denote* $([i^{1/n}]+1)^n - [i^{1/n}]$. *Then*

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$$f_n(i) = i + [i^{1/n}] + [i/p(n,i)] \quad \textit{for each } i \in N. \tag{4}$$

Proof. Choose $i \in N$. Then there is a unique $k \in N$ for which $k^n \leqslant i < (k+1)^n$. So $k \leqslant i^{1/n} < k+1$, and thus $k = [i^{1/n}]$. Also it is clear that $p(n,i) = (k+1)^n - k > 0$.

Since the positive integer $(k+1)^n - 1$ is not a perfect nth power, there exists $j_k \in N$ such that $f_n(j_k) = (k+1)^n - 1$. Since the function f_n is strictly increasing, and since obviously $k^n + k < (k+1)^n - 1 < (k+1)^n + (k+1)$—*i.e.*, $f_n(k^n) < f_n(j_k) < f_n((k+1)^n)$—it follows from (3) that $k^n < j_k < (k+1)^n$. Moreover, $f_n(j_k + 1) = (k+1)^n + 1$ since $(k+1)^n$ is a perfect nth power.

Clearly, $k^n < f_n(k^n) < f_n(j_k) = (k+1)^n - 1 < (k+1)^n < f_n(j_k+1) = (k+1)^n + 1 < f_n((k+1)^n) < (k+2)^n$. That is to say, there is exactly one perfect nth power—namely $(k+1)^n$—between $f_n(k^n)$ and $f_n((k+1)^n)$.

In the table below, we see the behavior of $f_n(i)$ for $k^n \leqslant i \leqslant (k+1)^n$.

i	k^n	k^n+1	...	j_k	j_k+1	...	$(k+1)^n$
$f_n(i)$	k^n+k	k^n+k+1	...	$(k+1)^n-1$	$(k+1)^n+1$	...	$(k+1)^n+k+1$

Note that as the argument i increases consecutively from k^n to $(k+1)^n$, the values $f_n(i)$ also increase consecutively except that $f_n(i)$ skips $(k+1)^n$. Thus for $k^n \leqslant s \leqslant t \leqslant (k+1)^n$

$$f_n(t) - f_n(s) = \begin{cases} t - s + 1 & \text{if } s \leqslant j_k \text{ and } t > j_k \\ t - s & \text{otherwise.} \end{cases} \tag{5}$$

By (5) we have that $f_n((k+1)^n) - f_n(j_k) = (k+1)^n - j_k + 1$. Then by substituting for $f_n((k+1)^n)$ and $f_n(j_k)$ and solving for j_k, we get

$$\begin{aligned} j_k &= (k+1)^n - k - 1 \\ &= \left([i^{1/n}] + 1\right)^n - [i^{1/n}] - 1 \\ &= p(n,i) - 1. \end{aligned} \tag{6}$$

Again using (5), we have that $f_n(i) = f_n(k^n) + i - k^n$ if $k^n \leqslant i \leqslant j_k$. Substituting for $f_n(k^n)$, we get that $f_n(i) = (k^n + k) + i - k^n = i + k$ and thus that

$$f_n(i) = i + [i^{1/n}] \quad \text{if } k^n \leqslant i \leqslant j_k. \tag{7}$$

Similarly we have that $f_n(i) = f_n(k^n) + i - k^n + 1$ if $j_k < i < (k+1)^n$. Thus, $f_n(i) = (k^n + k) + i - k^n + 1 = i + k + 1$ and

$$f_n(i) = i + [i^{1/n}] + 1 \quad \text{if } j_k < i < (k+1)^n. \tag{8}$$

By (6) if $k^n \leqslant i \leqslant j_k$ then $k^n \leqslant i < p(n,i)$, whence $0 < i/p(n,i) < 1$. It follows that

$$[i/p(n,i)] = 0 \quad \text{if } k^n \leqslant i \leqslant j_k. \tag{9}$$

Suppose on the other hand that $j_k < i < (k+1)^n$. Then $p(n,i) \leqslant i < (k+1)^n$ whence $1 \leqslant i/p(n,i) < (k+1)^n/p(n,i) = (k+1)^n/((k+1)^n - k)$. But since $n > 1$ and since $k > 0$ we see that $2k < (k+1)^2 \leqslant (k+1)^n$, and hence that $k < (k+1)^n/2$. Therefore, $i/p(n,i) < (k+1)^n/((k+1)^n - k) < (k+1)^n/((k+1)^n - (k+1)^n/2) = 2$. It follows that $1 \leqslant i/p(n,i) < 2$, and, hence, that

$$[i/p(n,i)] = 1 \quad \text{if } j_k < i < (k+1)^n \tag{10}$$

The theorem is now the immediate consequence of (7) and (8) together with (9) and (10).

Define a function to be *elementary* if it can be expressed by an explicit finite formula using only the operations of addition, subtraction, multiplication, division, roots, powers, and the greatest integer function. Define an injective sequence $\langle v_1, v_2, v_3, \dots \rangle$ to be *elementarily generated* if there exists an elementary function f such that $f(i) = v_i$ for every $i = 1, 2, 3, \dots$. By the *complement of a sequence* $t = \langle t_1, t_2, t_3, \dots \rangle$ of positive integers we mean the sequence obtained from $s = \langle 1, 2, 3, \dots \rangle$ by deleting from s all of the terms in t.

An easy cardinality argument shows that there exist subsequences of s which are not elementarily generated. For, while there are only countably many elementary functions there are uncountably many subsequences of s.

We believe the following questions to be open:

(i) For positive integers a and n, if $f(x) = ax^n$, then is the complement of $\langle f(1), f(2), f(3), \dots \rangle$ elementarily generated? Our theorem answers this question affirmatively for the cases where $a = 1$ and $n > 1$.

(ii) If $p(x)$ is a polynomial of positive degree with nonnegative integer coefficients, then is the complement of $\langle p(1), p(2), p(3), \dots \rangle$ elementarily generated?

(iii) Is each finite increasing sequence of positive integers elementarily generated?

(iv) Is the complement of a finite sequence of positive integers elementarily generated?

(v) Is the complement of an elementarily generated sequence of positive integers elementarily generated?

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REFERENCES

1. R. Honsberger, *Ingenuity in Mathematics*, Mathematical Association of America, 1970.
2. M. W. Ecker, Mathematical recreations: the fundamental counting principle, *Byte Magazine* 10 (1985), 425–428.

A Pessimistic Note on Fermat's Last Theorem

$$(3 + \sqrt{93})^3 + (3 - \sqrt{93})^3 = 12^3.$$

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