*Acknowledgment.* The author wishes to express his appreciation to the editors for their valuable suggestions.

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# An Improper Application of Green's Theorem

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The improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx \tag{1}$$

converges to $\pi/2$, but how is this value calculated? Since the antiderivative of $f(x) = \sin x/x$ cannot be expressed as a finite combination of elementary functions (see [**1**]), we must look beyond the Fundamental Theorem of Calculus. We present here a way to calculate (1) using only techniques covered in calculus, but we first present two standard calculations.

First, consider the Laplace transform $F$ of $\sin x/x$ :

$$F(s) = \int_0^\infty \frac{\sin x}{x} e^{-sx}\,dx.$$

As $s$ becomes large, the kernel $e^{-sx}$ converges rapidly to 0 as long as $x > 0$. Since $|\sin x/x| < 1$ for $x > 0$, we have $|F(s)| < 1/s$, which gives $\lim_{s\to\infty} F(s) = 0$.

Assuming that differentiation with respect to $s$ may be perfomed either before or after the integration, we have

$$\frac{dF}{ds} = -\int_0^\infty (\sin x)e^{-sx}\,dx = \frac{-1}{s^2+1}.$$

Hence $F(s) = -\arctan(s) + C$, where $C$ is a constant. But $\lim_{s\to\infty} F(s) = 0$, so $C$ must be $\pi/2$. Thus, since $C = F(0)$,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2.$$

For students with a solid background in integral calculus, the broad strokes of this calculation are easy to follow. We took a leap, though, when we differentiated with respect to $s$. This step, while justifiable, goes beyond topics covered in most calculus texts (conditions under which derivatives may be passed across integral signs are not given in [**3**], for example).

A second calculation of (1) is found in [**4**] (see pp. 278 and 189, Problems 6.14 and 6.15). This calculation starts with an improper *double* integral over the first quadrant:

 

$$\iint_{\mathbb{R}^+\times\mathbb{R}^+} e^{-xy}\sin x\,dA.$$

If we integrate first with respect to $y$, we get

$$\int_0^\infty\int_0^\infty e^{-xy}\sin x\,dy\,dx = \int_0^\infty \frac{\sin x}{x}\,dx.$$

On the other hand, integrating with respect to $x$ first gives

$$\int_0^\infty\int_0^\infty e^{-xy}\sin x\,dx\,dy = \int_0^\infty \frac{1}{1+y^2}\,dy.$$

It takes some work to prove, but changing the order of integration in the improper double integral does not change its value (see [**2**] for a very nice worksheet detailing the steps in the proof). Therefore,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1}{1+y^2}\,dy = \pi/2.$$

Improper double integrals, however, are not emphasized in most calculus texts.

Green's theorem, though, is a well-developed topic in calculus, and we use it to give a new calculation of (1). We first recall Green's theorem.

**Green's Theorem.** *Let $\Gamma$ be a positively-oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$, and suppose $D$ is the region enclosed by $\Gamma$. If $P$ and $Q$ are continuously differentiable on an open set containing $D$, then*

$$\int_\Gamma P(x, y)\,dx + Q(x, y)\,dy = \iint_D (Q_x(x, y) - P_y(x, y))\,dA.$$

For our calculation, we choose $Q = 0$ and search for $P$ such that $P(x, 0) = \sin x/x$. Looking at the double integral in Green's theorem, we see that we would like $P_y$ to be relatively easy to integrate. The Laplace transform calculation suggests that we try

$$P(x, y) = \begin{cases} \dfrac{e^{-xy}\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

This function is continuously differentiable on $\mathbb{R}^2$, and if $(x, y)$ is restricted to the first quadrant, then $P(x, y)$ converges to 0 as $(x, y)$ moves far from the origin. These two properties are essential in what follows.

Given a positive number $R$, define a positively-oriented square $\Gamma^R = \Gamma_1^R + \Gamma_2^R + \Gamma_3^R + \Gamma_4^R$ as in the figure. Green's theorem and Fubini's theorem tell us that

$$\int_{\Gamma^R} P(x, y)\,dx = -\iint_D P_y(x, y)\,dA = \int_0^R\int_0^R e^{-xy}\sin x\,dx\,dy.$$

The antiderivative of $e^{-xy}\sin x$ with respect to $x$ is $-(e^{-xy}(\cos x + y\sin x))/(1+y^2)$, so we have

$$\int_{\Gamma^R} P(x,y)\,dx = -\int_0^R \left(\frac{e^{-Ry}(\cos R + y\sin R)}{1+y^2} - \frac{1}{1+y^2}\right) dy$$
$$= \arctan(R) - \int_0^R \frac{e^{-Ry}(\cos R + y\sin R)}{1+y^2}\,dy. \qquad (2)$$

Since the line integral decomposes into a sum over the segments of $\Gamma^R$, (2) is equivalent to

$$\int_{\Gamma_1^R} P(x,y)\,dx + \int_{\Gamma_2^R} P(x,y)\,dx + \int_{\Gamma_3^R} P(x,y)\,dx + \int_{\Gamma_4^R} P(x,y)\,dx$$
$$= \arctan(R) - \int_0^R \frac{e^{-Ry}(\cos R + y\sin R)}{1+y^2}\,dy.$$

Natural parameterizations of the four segments that make up $\Gamma^R$ transform this into

$$\int_0^R \frac{\sin x}{x}\,dx = \arctan(R) + \int_0^R \frac{e^{-xR}\sin x}{x}\,dx - \int_0^R \frac{e^{-Ry}(\cos R + y\sin R)}{1+y^2}\,dy.$$

We next show that the last two integrals converge to 0 as $R \to \infty$. Then $\arctan(R)$ gives us the value $\pi/2$.

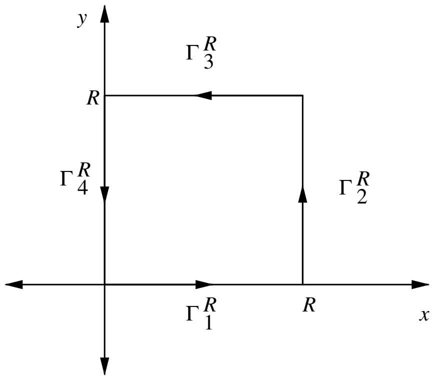


**Figure 1.** The closed curve $\Gamma^R$

We handle the last integral first. If $y \geq 0$, then $|y\sin(R) + \cos(R)| \leq y + 1$, so

$$\left|\int_0^R \frac{e^{-Ry}(\cos R + y\sin R)}{1+y^2}\,dy\right| \leq \int_0^R e^{-Ry}(y+1)\,dy$$
$$= \frac{-(R+1)e^{-R^2}+1}{R} - \frac{e^{-R^2}-1}{R^2},$$

and both of these terms go to 0 as $R \to \infty$.

 

For the other integral, since $|\sin x/x| < 1$ for $x > 0$, we get

$$\left|\int_0^R \frac{e^{-xR}\sin x}{x}\,dx\right| \le \int_0^R e^{-xR}\,dx = \frac{1-e^{-R^2}}{R},$$

which also converges to 0 as $R \to \infty$.

We note that these estimates are identical to those used in [**2**] to show that the order of integration makes no difference in the double integral. Thus, we can view this Green's Theorem calculation as a modification of the double integral method.

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# Partial Fractions by Substitution


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The standard method for finding the partial fraction decomposition for a rational function involves solving a system of linear equations. In this note, we present a quick method for finding the partial fraction decomposition of a rational function in the special case when the denominator is a power of a single linear or irreducible quadratic factor, that is, the denominator is either $(ax+b)^k$ or $(ax^2+bx+c)^k$ with $4ac > b^2$. For example, we note that substituting $t+2$ for $x$ and then expanding the numerator transforms

$$\frac{x^2+4x-3}{(x-2)^3} \quad \text{to} \quad \frac{t^2+8t+9}{t^3}.$$

Since this last expression splits into

$$\frac{1}{t} + \frac{8}{t^2} + \frac{9}{t^3},$$

it follows that our original function has

$$\frac{1}{x-2} + \frac{8}{(x-2)^2} + \frac{9}{(x-2)^3}$$

as its partial fraction decomposition. We observe that the numbers 9, 8, and 1 in the numerators of the decomposition could also have been obtained as the remainders by successive division of $x^2+4x-3$ by $x-2$. This method was considered by Kung [**4**] in this journal. Our substitution-expansion method avoids such repeated division