A Polynomial Taking Integer Values

ROBIN CHAPMAN
University of Exeter
EX4 4QE, United Kingdom

In **[2]** Sury proves that for integers $a_1 < a_2 < \cdots < a_n$, the expression $\prod_{n \geq i > j \geq 1}(a_i - a_j)\ (i-j)$ is also an integer. (The result follows immediately from the theory of Lie groups; the number turns out to be the dimension of an irreducible representation of $SU(n)$.) Sury gives an elementary but indirect proof, based on the stronger result that $\prod_{n \geq i > j \geq 1}(X^{a_i - a_j} - 1)\ (X^{i-j} - 1) \in \mathbb{Z}[X]$. A direct proof of the original result can be deduced from properties of the Vandermonde determinant and the fact that binomial coefficients are integral.

If we define $\Delta(a_1, a_2, \ldots, a_n) = \prod_{n \geq i > j \geq 1}(a_i - a_j)$, then our task is to show that $\Delta(1, 2, \ldots, n) = \prod_{i=1}^{n}(i-1)!$ divides $\Delta(a_1, a_2, \ldots, a_n)$ whenever $a_1 < a_2 < \cdots < a_n$ are integers. Since $\Delta(a_1, a_2, \ldots, a_n) = \Delta(a_1 + 1, a_2 + 1, \ldots, a_n + 1)$ we may assume that each $a_i \geq 0$. It is well known that $\Delta(a_1, a_2, \ldots, a_n)$ is the value of the *Vandermonde determinant*

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

(A careful proof of this fact can be found in section 2.9 of **[1]**; alternatively a good exercise is to obtain this formula by row and column operations and induction.) Applying elementary row operations to this determinant shows that if f_i are any monic polynomials of degree i, $1 \leq i \leq n-1$, then

$$\Delta(a_1, a_2, \ldots, a_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ f_1(a_1) & f_1(a_2) & f_1(a_3) & \cdots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & f_2(a_3) & \cdots & f_1(a_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(a_1) & f_{n-1}(a_2) & f_{n-1}(a_3) & \cdots & f_{n-1}(a_n) \end{vmatrix}.$$

In particular if we choose $f_i(a) = a(a-1)(a+2)\cdots(a-i+1)$, then for non-negative integers a, $f_i(a) = i!\binom{a}{i}$, so each entry in the i-th row of this determinant is divisible by $(i-1)!$. Hence $\Delta(a_1, a_2, \ldots, a_n)$ is divisible by $\prod_{i=1}^{n}(i-1)!$.

REFERENCES

1. David Sharpe, *Rings and Factorization*, Cambridge University Press, Cambridge, UK, 1987.
2. B. Sury, *An integral polynomial*, this MAGAZINE 68, 2 (1995), 134–135.