The Pearson and Cauchy-Schwarz Inequalities

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Recently, there has been renewed interest in the Cauchy-Schwarz inequality and particularly in its applications outside vector algebra [**1**]. The Cauchy-Schwarz inequality for vectors in $\mathbb{R}^n$ asserts that for any two vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ and $\mathbf{y} = \langle y_1, y_2, \ldots, y_n \rangle$ in $\mathbb{R}^n$, $(\mathbf{x} \cdot \mathbf{y})^2 \le \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$, where $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n (x_i y_i)$, $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n (x_i)^2}$, and $\|\mathbf{y}\| = \sqrt{\sum_{i=1}^n (y_i)^2}$.

For a sample of n data pairs $(x_1, y_1), (x_1, y_2), \ldots, (x_n, y_n)$, Pearson's coefficient of linear correlation is defined as (see [**2**] for example)

$$r = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}\sqrt{\sum_{i=1}^n (y_i - \overline{y})^2}},$$

where as usual $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and $\overline{y} = \frac{1}{n}\sum_{i=1}^n y_i$. Pearson's inequality, $|r| \le 1$, is immediate from the Cauchy-Schwarz inequality applied to the vectors $\mathbf{d}_x = \langle x_1 - \overline{x}, \ldots, x_n - \overline{x} \rangle$ and $\mathbf{d}_y = \langle y_1 - \overline{y}, \ldots, y_n - \overline{y} \rangle$.

The purpose of this note is to prove the converse.

Theorem 1. *For any two vectors* $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ *and* $\mathbf{y} = \langle y_1, y_2, \ldots, y_n \rangle$ *in* $\mathbb{R}^n$, *if* $|r| \le 1$, *then* $\left|\sum_{i=1}^n (x_i y_i)\right| \le \sqrt{\sum_{i=1}^n (x_i)^2}\sqrt{\sum_{i=1}^n (y_i)^2}$.

Proof. We apply Pearson's coefficient formula to two new vectors

$$\mathbf{x}^* = \left\langle \frac{-x_1}{\sqrt{2}}, \frac{-x_2}{\sqrt{2}}, \ldots, \frac{-x_n}{\sqrt{2}}, \frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, \ldots, \frac{x_n}{\sqrt{2}} \right\rangle$$

and

$$\mathbf{y}^* = \left\langle \frac{-y_1}{\sqrt{2}}, \frac{-y_2}{\sqrt{2}}, \ldots, \frac{-y_n}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}, \ldots, \frac{y_n}{\sqrt{2}} \right\rangle$$

in $\mathbb{R}^{2n}$. Since $\overline{x^*} = \overline{y^*} = 0$,

$$\left| \frac{\sum_{i=1}^n (x_i^* y_i^*)}{\sqrt{\sum_{i=1}^n (x_i^*)^2}\sqrt{\sum_{i=1}^n (y_i^*)^2}} \right| \le 1$$

implies that

$$\left| 2\sum_{i=1}^n \frac{(x_i y_i)}{2} \right| \le \sqrt{2\sum_{i=1}^n \frac{(x_i)^2}{2}}\sqrt{2\sum_{i=1}^n \frac{(y_i)^2}{2}},$$

from which the desired result follows. ■

Thus Pearson's inequality is equivalent to the Cauchy-Schwarz inequality.

References

1. W. M. Dunn III, A quick proof that the least squares formulas give a local minimum, *College Math. J.* **36** (2005) 64–65.
2. R. Larson and B. Farber, *Elementary Statistics Picturing the World*, 3rd ed., Prentice Hall, 2006.