The Gaps Between Consecutive Binomial Coefficients

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Among all combinatorial quantities, the binomial coefficients are unique. They are simple in concept and derivation, yet they appear in almost all the combinatorial identities and seem to give rise to an endless variety of problems. In this note we investigate one such problem which we believe to be new. Put in a simple way, the problem is to determine the smallest and largest "gaps" between consecutive binomial coefficients in any row of Pascal's triangle. More precisely, for positive integers n and k with $n \geqq 2$ and $1 \leqq k \leqq n$, we define

$$d_k(n) = \left| \binom{n}{k} - \binom{n}{k-1} \right|$$

and we let $d_m(n) = \min_{1 \leqq k \leqq n} d_k(n)$, and $d_M(n) = \max_{1 \leqq k \leqq n} d_k(n)$. Our purpose is to solve the following problems:

P1. Determine $d_m(n)$ and all k such that $d_k(n) = d_m(n)$.

P2. Determine $d_M(n)$ and all k such that $d_k(n) = d_M(n)$.

We start out with P1, which is the easier one. Due to symmetry, it clearly suffices to consider $d_k(n)$ for $1 \leqq k \leqq [(n+1)/2]$. Since

$$\binom{n}{k} - \binom{n}{k-1} = \frac{n!(n-2k+1)}{k!(n-k+1)!} = 0$$

if and only if $n - 2k + 1 = 0$, we see that $d_m(n) = 0$ if and only if n is odd and $d_k(n) = 0$ if and only if $k = (n+1)/2$. When n is even we shall show that $d_m(n) = n - 1$ except when $n = 4$ in which case direct inspection reveals that $d_m(4) = 2$. In fact, we shall show that in general if we define $d'_m(n) = \min\{d_k(n) | 1 \leqq k \leqq n,\ k \neq (n+1)/2\}$ (so that the aforementioned trivial fact that $d_m(n) = 0$ when n is odd is excluded), then $d'_m(n) = n - 1$ holds for all $n \neq 4$.

THEOREM 1. *For all $n \geqq 2$, except when $n = 4$, $d'_m(n) = n - 1$. Furthermore, if $n \neq 6$, then $d_k(n) = n - 1$ if and only if $k = 1$ or n, and if $n = 6$, then $d_k(6) = 5$ if and only if $k = 1, 3, 4,$ or 6.*

Proof. Since the cases when $n \leqq 6$ can be verified directly by inspection, we assume that $n > 6$ and use induction. Examining the appropriate rows in Pascal's triangle reveals that the theorem holds for $n = 7$ and 8. Due to symmetry and the fact that $d_1(n) = d_n(n) = n - 1$ we need only to show that $d_k(n) > n - 1$ for all k such that $2 \leqq k \leqq [n/2]$ where $n \geqq 9$. By Pascal's identity, we have:

$$\binom{n}{k} - \binom{n}{k-1} = \left\{ \binom{n-1}{k} + \binom{n-1}{k-1} \right\} - \left\{ \binom{n-1}{k-1} + \binom{n-1}{k-2} \right\}$$

$$= \left\{ \binom{n-1}{k} - \binom{n-1}{k-1} \right\} + \left\{ \binom{n-1}{k-1} - \binom{n-1}{k-2} \right\}. \tag{1}$$

If n is odd, then by the induction hypothesis, we have

$$\binom{n-1}{k} - \binom{n-1}{k-1} \geqq n-2 \quad \text{and} \quad \binom{n-1}{k-1} - \binom{n-1}{k-2} \geqq n-2.$$

Hence from (1) we obtain

$$\binom{n}{k} - \binom{n}{k-1} \geqq 2(n-2) = (n-1) + (n-3) > n-1. \tag{2}$$

If n is even then the above argument still holds except when $k = [n/2]$ in which case

$$\binom{n-1}{k} - \binom{n-1}{k-1} = 0.$$

By a second application of Pascal's identity we obtain from (1)

$$\begin{aligned} \binom{n}{k} - \binom{n}{k-1} &= \binom{n-1}{k-1} - \binom{n-1}{k-2} \\ &= \left\{ \binom{n-2}{k-1} - \binom{n-2}{k-2} \right\} + \left\{ \binom{n-2}{k-2} - \binom{n-2}{k-3} \right\}. \end{aligned} \tag{3}$$

Since $n-2$ is even we have, by the induction hypothesis,

$$\binom{n-2}{k-1} - \binom{n-2}{k-2} \geqq n-3 \quad \text{and} \quad \binom{n-2}{k-2} - \binom{n-2}{k-3} \geqq n-3.$$

Hence from (3), we obtain

$$\binom{n}{k} - \binom{n}{k-1} \geqq 2(n-3) = (n-1) + (n-5) > n-1. \tag{4}$$

By (2) and (4) our proof is complete.

Now we turn to P2 the answer of which is somewhat less obvious than that of P1. The complete answer to this problem is contained in the next

THEOREM 2. *Let* $\tau = (1/2)(n + 2 - \sqrt{n+2})$. *Then*

$$d_M(n) = \binom{n}{[\tau]} - \binom{n}{[\tau]-1}.$$

Furthermore, $d_k(n) = d_M(n)$ *if and only if*

$$k = \begin{cases} [\tau] \text{ or } n - [\tau] + 1 \text{ if } \tau \notin \mathbb{Z} \\ \tau - 1, \tau, n - \tau + 1 \text{ or } n - \tau + 2 \text{ if } \tau \in \mathbb{Z} \end{cases}$$

where $\mathbb{Z}$ *denotes the set of integers.*

Proof. Due to symmetry we may again assume that $k \leqq [n/2]$. Direct computations show that

$$\left\{ \binom{n}{k} - \binom{n}{k-1} \right\} - \left\{ \binom{n}{k-1} - \binom{n}{k-2} \right\} = \frac{n!\sigma}{k!(n-k+2)!}$$

where

$$\begin{aligned}\sigma &= (n-k+2)(n-2k+1) - k(n-2k+3)\\ &= (n-2k+1)(n-2k+2) - 2k\\ &= (2k)^2 - (2n+4)(2k) + (n+1)(n+2)\\ &= \{2k-(n+2)\}^2 - (n+2)\\ &= \{2k-(n+2)+\sqrt{n+2}\}\{2k-(n+2)-\sqrt{n+2}\}\\ &= 4(k-\tau)\{k-\tfrac{1}{2}(n+2+\sqrt{n+2})\},\end{aligned}$$

where $\tau = (1/2)(n+2-\sqrt{n+2})$. Since $k \leqq [n/2] < (1/2)(n+2+\sqrt{n+2})$ we conclude that

$$\binom{n}{k} - \binom{n}{k-1} \geqq \binom{n}{k-1} - \binom{n}{k-2}$$

if and only if $k \leqq \tau$ with equality holding if and only if $\tau \in \mathbb{Z}$. The statement of our theorem now follows immediately.

To illustrate Theorem 2, we consider an example.

Example. When $n = 13$, $\tau = \frac{1}{2}(15 - \sqrt{15}) \notin \mathbb{Z}$ and $[\tau] = 5$. Thus the largest gap in the 13th row of Pascal's triangle occurs twice: once between $\binom{13}{4}$ and $\binom{13}{5}$, and the other between $\binom{13}{8}$ and $\binom{13}{9}$. These gaps have absolute value $\binom{13}{5} - \binom{13}{4} = 1287 - 715 = 572$. When $n = 14$, $\tau = 6 \in \mathbb{Z}$. Thus the largest gaps in the 14th row of Pascal's triangle occur four times: between $\binom{14}{4}$ and $\binom{14}{5}$; $\binom{14}{5}$ and $\binom{14}{6}$; $\binom{14}{8}$ and $\binom{14}{9}$; and between $\binom{14}{9}$ and $\binom{14}{10}$. The absolute value of these gaps is $\binom{14}{5} - \binom{14}{4} = 2002 - 1001 = 1001$.

Theorem 2 also indicates an interesting fact regarding consecutive binomial coefficients which form an arithmetic progression. It is well known [1, p. 54] that no four consecutive binomial coefficients can form an arithmetic progression. Our result in Theorem 2 implies that there are infinitely many triples of consecutive binomial coefficients which form an arithmetic progression. In fact, we have:

COROLLARY. *If $n > 2$ such that $n+2$ is a perfect square, then $\binom{n}{\tau-2}$, $\binom{n}{\tau-1}$, and $\binom{n}{\tau}$ form an arithmetic progression where $\tau = (1/2)(n+2-\sqrt{n+2})$. Furthermore, the common difference of this arithmetic progression yields the largest gap in the nth row of Pascal's triangle.*

In closing, we point out that if, in the left half of each row in Pascal's triangle, we use a dot to represent the larger one of the binomial coefficients whenever a largest gap occurs and connect the dots in consecutive rows with line segments, then we would obtain a chain which has some kind of zigzag pattern and which has a "diamond" whenever $n > 2$ is such that $n+2$ is a perfect square. We leave it to the readers to do the actual drawing.

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