Trick or Technique?

Michael Sheard

Michael Sheard (SheardM@rhodes.edu) is the chair of the Department of Mathematics and Computer Science at Rhodes College in Memphis, TN, a position he recently took on after 22 years at St. Lawrence University in Canton, NY. His main area of research is set theory and logic, with a special emphasis on some topics that lie right on the boundary of mathematics and philosophy. When not engaged in professional activities, he can often be found outdoors with his family, canoeing, hiking, and especially birding.

A perfectly ordinary problem from the integrals section of a calculus course is

$$\int \frac{\ln(x)}{x}\,dx.$$

Most students will recognize it immediately as an easy example of integration by substitution: $u = \ln(x)$. But sometimes a funny thing can happen, especially if the problem appears on an assignment after the class has covered a few more integration techniques. Some students may try to solve the problem by integration by parts: $u = \ln(x)$, $dv = \frac{1}{x}\,dx$. Then $du = \frac{1}{x}\,dx$, $v = \ln(x)$, and

$$\int \frac{\ln(x)}{x}\,dx = \ln(x)\cdot\ln(x) - \int \frac{\ln(x)}{x}\,dx.$$

Students who have made it this far are likely to drop the approach, realizing that they have "simplified" the original problem into another copy of itself. But a few students just might notice that they can now finish the solution by algebra alone:

$$2\int \frac{\ln(x)}{x}\,dx = \ln(x)\cdot\ln(x) + C$$
$$\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}\ln(x)^2 + C.$$

(The constant of integration appears, of course, because the two occurrences of the indefinite integral are only equal up to an additive constant.)

This is a cute trick. It feels like magic, because it seems that at no time was any "rule" for integrals actually invoked. More complex (and genuinely useful) variations of this trick appear in many calculus textbooks, toward the tail end of the section on integration by parts. The classic example is some variation of $\int e^x \sin(x)\,dx$, which involves *two* integrations by parts before algebra is applied (see, for example, [**2**], pp. 491–493). Another application is the derivation of reduction formulas for the integrals of powers of trigonometric functions. But this simplest version—one application of integration by parts, with no further integrations or use of identities—seems like it might be a useful method on its own. Let's call it the *one-step algebra trick*.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA

Are there other applications of the one-step algebra trick? And, if so, why isn't it taught as an actual technique of integration? The goal of this note is to see if there are enough other uses of this trick to make it a technique worth teaching—and also to have some fun computing integrals in unexpected ways. Afterward, we will look briefly at some of the more complicated variations.

The one-step trick

Remarkably, it is easy to find several simple—if thoroughly unnecessary—applications of the one-step algebra trick. For example, let's compute $\int x^{17}\,dx$ by parts. Let $u = x^{17}$, $dw = dx$, so $du = 17x^{16}$ and $w = x$, and then

$$\begin{aligned}\int x^{17}\,dx &= x^{17}\cdot x - \int x\cdot 17x^{16}\,dx\\ &= x^{18} - 17\int x^{17}\,dx.\end{aligned}$$

So

$$\int x^{17}\,dx = \frac{1}{18}x^{18} + C$$

and we have dispensed with the power rule for integrals!

In fact, the one-step algebra trick can be applied to this same integral in many different ways. To compute $\int x^n\,dx$, pick any m and k such that $n = m + k$ with $k \neq -1$. Then let $u = x^m$ and $dv = x^k\,dx$, and apply integration by parts. Try it—you will need the power rule for x^k, but not for x^n.

For an even sillier application, consider this computation:

$$\begin{aligned}\int e^{2x}\,dx = \int e^x\cdot e^x\,dx &= e^x\cdot e^x - \int e^x\cdot e^x\,dx\\ 2\int e^x\cdot e^x\,dx &= e^x\cdot e^x + C\\ \int e^{2x}\,dx &= \frac{1}{2}e^{2x} + C.\end{aligned}$$

This time we have used integration by parts to circumvent integration by substitution.

Now let's look for a general pattern. Suppose we have a problem which is solvable by the one-step algebra trick: some integral of the form $\int f(x)g'(x)\,dx$. We apply integration by parts to get

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

If the trick is to work, we need to be able to rewrite the last term as a multiple of the left-hand side:

$$\int g(x)f'(x)\,dx = h(x)\int f(x)g'(x)\,dx.$$

Of course, we can always define a function $h(x)$ to make this work, but if this is to be a useful elementary integration technique, the transformation from $\int g(x)f'(x)\,dx$

to $h(x)\int f(x)g'(x)\,dx$ should be an obvious and straightforward one. In elementary integral calculus, the only "factoring out" which is permitted is the factoring of constants—indeed, instructors typically have to discourage students from factoring expressions *other* than constants out of integrals. So we expect to be able to use this trick only for $h(x) = K$ for some constant $K \neq -1$. Thus $g(x)f'(x) = Kf(x)g'(x)$. Now let's apply the most basic technique for solving differential equations, separation of variables:

$$\frac{f'(x)}{f(x)} = K\frac{g'(x)}{g(x)}.$$

Integrate both sides:

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\,dx &= K\int \frac{g'(x)}{g(x)}\,dx\\ \ln|f(x)| &= K\cdot\ln|g(x)| + C\\ |f(x)| &= |g(x)|^K \cdot e^C,\end{aligned}$$

and then letting $C_1 = \pm e^C$,

$$f(x) = C_1(g(x))^K$$

with $K \neq -1$.

The moral of this computation is that the one-step algebra trick can only be used on integrals of the form $\int (g(x))^K g'(x)\,dx$, which is to say integrals which can *already* be integrated by ordinary substitution and the power rule. Now we can see why the one-step algebra trick is not taught as a technique. Any integral which can be computed by the one-step algebra trick can already be computed by a simpler and (probably) more obvious method.

One-step variations

Notice that after the initial integration by parts, the remaining integrand need not be a simple scalar multiple of the original integrand, but instead could be a sum of the original and another function which is easy to integrate, possibly by means of some identity. A nice example is provided by $\int \cos^2(x)\,dx$:

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \cos(x)\cdot\cos(x)\,dx\\ &= \cos(x)\cdot\sin(x) + \int \sin(x)\cdot\sin(x)\,dx\\ &= \cos(x)\cdot\sin(x) + \int (1-\cos^2(x))\,dx\\ &= \cos(x)\cdot\sin(x) + x - \int \cos^2(x)\,dx.\end{aligned}$$

Now by algebra:

$$\int \cos^2(x)\,dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C.$$

© THE MATHEMATICAL ASSOCIATION OF AMERICA

Unlike our earlier examples, this is actually a reasonable alternative to the usual method for integrating $\cos^2 x$ given in most textbooks, which invokes the half-angle identity $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ to obtain $\frac{1}{2}x + \frac{1}{4}\sin(2x) + C$. (Our answer is the same, via a double-angle identity.) The half-angle and double-angle identities are rarely encountered in the high school or undergraduate curriculum, except specifically for the integrals of $\cos^2 x$ and $\sin^2 x$, so, in my experience, students find them easy to forget. Our method works even if the identities are forgotten. This same pattern can be followed directly to compute $\int \sin^2(x)\,dx$ (of course), $\int \cosh^2(x)\,dx$, and $\int \sinh^2(x)\,dx$.

But how general is it? Following the approach used earlier, we can see that this more general method computes $\int f(x)g'(x)\,dx$ in situations in which $g(x)f'(x) = a(x) + bf(x)g'(x)$, with $b \neq -1$. We also need $a(x)$ to be easy to integrate.

Again proceeding formally, we can try to solve for the required relationship between $f(x)$ and $g(x)$. Rewrite the equation as

$$f'(x) - \frac{bg'(x)}{g(x)}f(x) = \frac{a(x)}{g(x)},$$

which is a first-order linear nonhomogeneous differential equation for $f(x)$. By the usual technique from a beginning differential equations course ([**3**]), the solution is

$$f(x) = g(x)^b \int \frac{a(x)}{g(x)^{b+1}}\,dx.$$

As a relationship between f and g, this is fairly daunting. The prototypical examples work out correctly, of course: setting $a = -1$, $b = 1$, $g(x) = \sin(x)$ yields $f(x) = \cos(x)$. In general, though, choosing even simple-looking functions for $g(x)$ and $a(x)$ and small values for b yields either trivial integrals or else remarkably complicated beasts. For example, making $g(x)$ something as simple as $x^2 + x$ while $a = -1$ and $b = 1$ produces the tedious

$$\int \left(2x + 1 + 2(x^2 + x)\ln\left(\frac{x}{x+1}\right)\right)(2x+1)\,dx,$$

which begs to be expanded out and then integrated in little pieces. Occasionally you may stumble upon a pleasing possibility—one reasonable example comes from taking $a(x) = b = 1$ and $g(x) = \tan(x)$ to get $\int (1 + x\tan x)\sec^2(x)\,dx$, which is a nice problem of intermediate difficulty for a beginning student—but such instances are rare.

The two-step trick

A full analysis of the algebra trick with *two* applications of integration by parts quickly becomes unmanageable, so let's just take a quick look at the most basic pattern. Since we need to integrate by parts twice, the notation is simplified if we begin by writing one of the parts as a second derivative. Then:

$$\int f(x)g''(x)\,dx = f(x)g'(x) - \int f'(x)g'(x)\,dx.$$

In the simplest circumstance, the parts for the second integration are already on display: $u = f'(x)$, $dv = g'(x)\,dx$. Then

$$\int f(x)g''(x)\,dx = f(x)g'(x) - f'(x)g(x) + \int f''(x)g(x)\,dx.$$

Now the algebra trick works if $f''(x)g(x) = kf(x)g''(x)$, or

$$\frac{f''(x)}{f(x)} = k\frac{g''(x)}{g(x)},$$

where $k \neq 1$.

For the simplest possible case, consider what happens if $\frac{f''(x)}{f(x)}$ is a constant (and so $\frac{g''(x)}{g(x)}$ is a different constant). Familiar functions with this property include $e^{\alpha x}$, $\cosh(\alpha x)$, $\sinh(\alpha x)$, $\cos(\alpha x)$, and $\sin(\alpha x)$. We can mix and match these functions in the roles of f and g to get integrals computable by a two-step algebra trick: out comes $\int e^{\alpha x}\sin(\beta x)\,dx$ (the classic example of textbook fame), along with odd-looking possibilities like $\int e^{\alpha x}\cosh(\beta x)\,dx$. One particularly interesting class of integrals that falls out of this box is the product of two sine and/or cosine functions, such as $\int \cos(mx)\cos(nx)\,dx$, which pop up in an important way in the development of Fourier series. As Nicol [**1**] has suggested, this approach can relieve the pressure to remember the relevant trigonometric identies. In the case that $\frac{f''(x)}{f(x)}$ is a not constant, things quickly become too messy to qualify as a useful technique. And, of course, we can complicate matters much more by applying an identity to rewrite the integrand after either of the two transformations by parts.

In short, it seems that the traditional wisdom embodied in the textbooks is correct. These tricks are useful in special cases, but the opportunities are too infrequent to warrant teaching them as a fundamental technique.

Acknowledgments. I want to thank the students of Math 136 for implicitly suggesting the topic of this paper.

References

1. S. J. Nicol, Integrals of products of sine and cosine with different arguments, this JOURNAL **24** (1993) 158–160.
2. J. Stewart, *Single Variable Calculus*, 6th ed., Thomson Brooks/Cole, Belmont CA, 2008.
3. D. G. Zill, *A First Course in Differential Equations with Modeling Applications*, 8th ed., Thomson Brooks/Cole, Belmont CA, 2005.

> So, what is mathematics, really? I think the essence of it is the search for pattern, for structure and regularity, and for connections between seemingly unrelated objects, whether "real" or abstract. In this sense it is quite akin to art, to music in particular.
>
> —Ali Maor, *The Pythagorean Theorem*, p. 208

© THE MATHEMATICAL ASSOCIATION OF AMERICA