The Symmedian Point: Constructed and Applied

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During my first year as a graduate student in physics at Yale, I worked part-time for the Navy. One of my tasks was to analyze classified data from a test the Navy was conducting. In the Navy’s scenario, enemy airplanes were parachuting mines into a harbor, putting it out of commission until the mines were cleared, that is, exploded. The problem was to find the mines. If left too long they would sink into the mud on the bottom of the harbor and become almost impossible to locate. The Navy’s idea was to build three observation stations around the harbor and equip them with high power telescopes. The three observers would track the descending parachute and record its angle from the station just as it hit the surface of the water. The three angles would then determine just where the mine landed (see Figure 1).

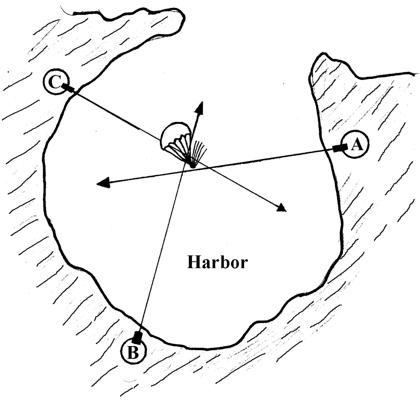

Figure 1. Sighting stations around a harbor.

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I had to perform the necessary calculations. In those days without computers (1951), in order to get the necessary precision one had to use large log tables. It was slow, difficult, boring, and easy to make mistakes. It was also hard to know when you had made a mistake. The first step was to calculate the intersection of the 3 sighting lines. I would then plot these intersection points on a piece of graph paper and see if they made any sense. The three lines never crossed at a single point and sometimes were far apart. The Navy wanted your best guess of where to start looking. 'Best guess' was defined as the point where the sum of the squares of the perpendicular distances from the bearing lines was a minimum (see Figure 2).

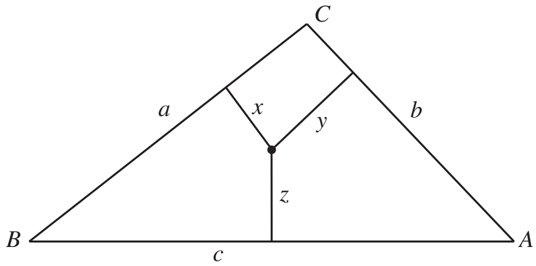

Figure 2. The Navy's best-guess: minimize $x^2 + y^2 + z^2$.

This took considerably more work. Eventually, I would plot this point on my graph paper and see if it was reasonable. This made finding mistakes much easier. I found that the eye-brain system could, with a little practice, guess quite well where the point should be located. I tried to convince my boss to let me just guess and skip the calculation. His response was "Nice try." No, the Navy wanted the full calculation.

It bothered me that my guesses were almost as good as the full calculation. Then I realized that there might be a geometric solution that the eye-brain system was sensing. What stood out was that the least squares point was always closest to the shortest side of the triangle formed by the 3 intersecting bearing lines, and next closest to the second shortest side of the triangle. I went back and took a look at the calculations in detail to see if there was a clue in this direction. When I compared the length of the perpendicular distance to the length of the corresponding side of the triangle, the ratios of the sides to the perpendicular distances, agreed exactly to the limit of the accuracy of my data. This relationship held for all of the triangles that I had calculated. It appeared to be a universal rule. After some experimentation I found a graphical construction that took advantage of this relationship.

It turns out that the point where the sum of the squares of the perpendicular distances from the three sides of the triangle is a minimum is well-known. It's the *symmedian* point. (In Britain and France, it is the Lemoine point; in Germany, Grebe's point.) It is considered a gem of modern plane geometry. Ross Honsberger [**1**] devotes 25 pages to it. (For its early history see [**3, 4**].) The symmedian is the sixth triangle center in Clark Kimberling's *Encyclopedia* [**2**]. Furthermore, it has the property that the ratios of the perpendicular distances to the sides of the triangle are constant, as I had empirically discovered.

My boss bought my construction and I was allowed to use it. It cut the time to one third of the original time and was much more error free. It also made the job, kind of fun.

My construction is illustrated in Figure 3. One begins by constructing squares, one on each side of the given triangle ABC. The sides of these squares that are parallel

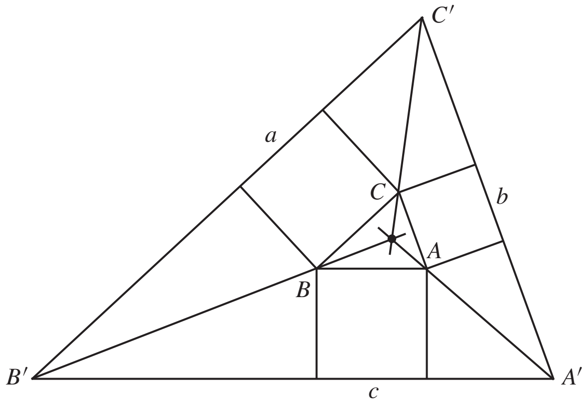

Figure 3. Construction of the symmedian.

to the sides of the given triangle, namely a, b, and c, are extended to form a second triangle, $A'B'C'$, similar to the given triangle. These two triangles have the same symmedian. The lines through the corresponding vertices of the two triangles meet at the symmedian point. This construction appears to be new.

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Summary. A post-WWII assignment from the U.S. Navy analyzing data from a test of a harbor mine-detecting algorithm leads to a new construction of the symmedian point.

References

1. R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Mathematical Association of America, Washington DC, 1995.
2. C. Kimberling, The Encyclopedia of Triangle Centers; available at `http://faculty.Evansville.edu/ck6/encyclopedia/ETC.html`.
3. J. Mackay, Early history of the symmedian point, *Proceedings of the Edinburgh Math Soc.* **XI** (1892) 92.
4. ———, Symmedians of a triangle and their concomitant circles, *Proceedings of the Edinburgh Math. Soc.* **XIV** (1895) 37–103.

We can say that Gödel made one of Hilbert's dreams come true, even though he crushed the other dream.

—from *Roads to Infinity, The Mathematics of Truth and Proof* by John Stillwell, reviewed on page 160