It follows that

$$\frac{R^2-x^2}{R+x}=\frac{2(a_1+a_2+\cdots+a_n)/n+\left(a_1^2+a_2^2+\cdots+a_n^2\right)/nx}{R/x+1} \tag{26}$$

where we have divided numerator and denominator by x. As x tends to infinity we note that $\lim R/x = 1$ from the calculation

$$\frac{R}{x}=\sqrt{\frac{1}{n}\left[\left(\frac{a_1}{x}+1\right)^2+\left(\frac{a_2}{x}+1\right)^2+\cdots+\left(\frac{a_n}{x}+1\right)^2\right]}.$$

Hence from (25) and (26) we have

$$\lim(R-x)=\lim\frac{R^2-x^2}{R+x}=\frac{2(a_1+a_2+\cdots+a_n)/n+0}{1+1}=A(a_1,a_2,\ldots,a_n),$$

as was claimed in (16).

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The Fifteen Billiard Balls— a Case Study in Combinatorial Problem Solving

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Balls bearing the numbers from 1 to 15 are on a billiard table. The object of the "game" is to knock all fifteen off the table (into the pockets), where any one of the fifteen balls can be knocked off *first*, but thereafter the next ball to go must be numbered consecutively to one which is already pocketed. Thus if "3" is the first to go, the next one to be removed can be either "2" or "4". If "3" and "4" are off the table, the next one to go can be either "2" or "5". If "3", "4", "2", and "1" are gone, the next one to go *must* be "5", and thereafter "6", then "7", etc., up to "15". (We do not regard "1" and "15" as adjacent!) The question is: *how many different sequences are permitted for removing all fifteen balls from the table*?

The permitted sequences are all permutations of the numbers from 1 to 15, so $15! = 1,307,674,368,000$ is a trivial upper bound on the number of permitted sequences. To get a tighter bound, we observe that while the first ball to go can be any of fifteen, each turn thereafter is a choice of *at most two*; and the very last to go is the only one left, so it is a "choice" of 1. This gives $15 \times 2^{13} \times 1 = 122,880$ as a more realistic upper bound. But how to account for the "sometimes 2, sometimes 1" nature of those intermediate cases?

The brute force solution

Suppose the first ball to go has the number k. If $k = 1$ or $k = 15$, there is only one way to complete the sequence. If $k = 2$, the remaining sequence will be ",3,4,5,6,...,14,15," except that ball number 1 can be inserted at any of the positions in the sequence indicated by a comma, from before 3 to after 15, for a total of 14 *different sequences*. By symmetry, we have the same result if the initial ball has the number $k = 14$.

For the general case, if k is the number of the first ball to go off the table, we have the higher numbers, which must be removed in the relative order $k+1, k+2, \ldots, 14, 15$; and the lower numbers, which must be removed in the relative order $k-1, k-2, \ldots, 2, 1$. The number of ways of interspersing these two subsequences is then $\binom{14}{k-1}$, since we must designate which $k-1$ of the fourteen remaining turns (after the first ball has been removed) will involve *lower* numbered balls, and that designation uniquely specifies the rest of that sequence, since there is a unique relative order among the lower-numbered balls, and a unique relative order among the remaining, higher-numbered balls. Hence *the total number of permitted sequences is*:

$$1 + 14 + \binom{14}{2} + \binom{14}{3} + \binom{14}{4} + \cdots + \binom{14}{14} = \sum_{j=0}^{14} \binom{14}{j} = (1+1)^{14} = 2^{14} = 16{,}384.$$

The simple form of the answer (it would have been 2^{n-1} if we had started with n billiard balls) suggests that there should be a much easier way of arriving at it.

The simple solution

In games (and puzzles) which consist of a finite number of **moves**, where each move involves a "decision" constrained by the rules of the game, it is typically the case that the analysis of the game is simplified by starting with the *last* move (the "winning"—or "losing"—move) and working backward.

In our billiard game, the very last ball to go in must be either "1" or "15", a binary "choice" (if we are running the videotape of this game in reverse!). If the last ball is "1", its predecessor must have been either "2" or "15", again a binary choice. (Similarly, if the last ball was "15", its predecessor was either "1" or "14".) In fact, as we view the game in *reverse*, we see that at each turn there is a binary choice: either the highest or the lowest numbered ball off the table will be the "next" to reappear. And this proceeds all the way back to where we see only one ball (the first ball) still on the table, which will be the unique "choice" for the "final" stage of our reverse process. So the number of possible sequences is trivially 2^{14} (or 2^{n-1}, if we had started with n billiard balls).

A simple "forward" solution

Now that we know our problem not only has a simple answer, but a simple way of arriving at it "in reverse", we can look for a simple "forward" solution. There are 14 **transitions** from one ball to the next as we remove all 15 balls from the table. If the transition is to a higher-numbered ball, let us represent it by $+$; if to a lower-numbered ball, by $-$. There are 2^{14} sequences of $+$'s and $-$'s of length 14, and we can show that they correspond precisely to the allowed sequences of billiard balls. For suppose that there are m minuses, and therefore $14 - m$ plusses. Then the first ball to go off the table had to bear the number $k = m + 1$, because the minuses in the sequence correspond to transitions to balls numbered lower than k, and the plusses to balls numbered higher than k. Moreover, starting at k, all the sequences of $m = k - 1$ minuses and $14 - m$ plusses precisely correspond to the distinct ways which are allowed to complete the sequence.

Note that this is really a restatement of the "brute force" (forward) solution, but with a simplified way of counting to arrive at 2^{14}. Reading the 2^{14} sequences of $+$'s and $-$'s *backward*, we have an obvious model for the "simple" (reverse) solution, where "$+$" means "remove a ball at the high end" and "$-$" means "remove a ball at the low end", in progressing from the fifteenth turn back to the second turn.

A more general problem

How many different ways (sequences), $s(t)$, are there to remove the first t of the fifteen balls from the billiard table, subject to our previous rules, for $1 \leqslant t \leqslant 15$? We know that $s(1)=15$ and $s(15)=2^{14}$. It turns out that

$$s(t)=(16-t)\cdot 2^{t-1}, \quad \text{for } 1 \leqslant t \leqslant 15.$$

(If we had started with n balls on the table, and the same basic rules, we would have $s(t)=(n+1-t)\cdot 2^{t-1}$, for $1 \leqslant t \leqslant n$.) Proving this formula is one of the harder ways to solve the original problem, but it is certainly doable.

Imagine a snake consisting of 15 segments numbered consecutively from *head* (#1) to *tail* (#15), where the segment numbers correspond to the billiard ball numbers. For the sequences of length t, we visualize the snake as having swallowed the last $t-1$ segments from its tail end, so that segments $1, 2, 3, \ldots, t-1$ now coincide with segments $15-t+1, 15-t+2, 15-t+3, \ldots, 15$, respectively. (For $t>7$, the snake is looped through itself more than once!) From any of the $16-t$ "distinct" segments, we can exactly represent the billiard sequences of length t by all possible strings of $t-1$ $+$'s and $-$'s, leading to $(16-t)\cdot 2^{t-1}$ as the total number of such sequences.

For $t=1$, we merely select one of the 15 distinct segments. For $t=2$, there is an identification of segment 1 with segment 15. From any of the other 13 segments, $+$ indicates that the second term of the sequence is the next higher number, while $-$ indicates that it is the next lower number. From segment 1/15, $+$ gives the sequence 1,2 while $-$ gives the sequence 15,14. For $t=3$, there is an identification of segment 1 with segment 14, and of segment 2 with segment 15. For any starting number from 3 to 13, inclusive, each of the patterns $++$, $+-$, $-+$, $--$ yields a different sequence of length three, in a normal way. TABLE 1 shows the unique interpretation of these four patterns from the starting values 1/14 and 2/15. The situation for $t=3$ is similarly summarized in TABLE 2. The "snake" for this case is illustrated in FIGURE 1. More generally, starting points for sequences are "distinct" if and only if they are distinct modulo $16-t$. The reader is invited to fill in the remaining details.

Start \ Pattern	$++$	$+-$	$-+$	$--$
1/14	1,2,3	14,15,13	14,13,15	14,13,12
2/15	2,3,4	2,3,1	2,1,3	15,14,13

TABLE 1

Start \ Pattern	$+++$	$++-$	$+-+$	$+--$
1/13	1,2,3,4	13,14,15,12	13,14,12,15	13,14,12,11
2/14	2,3,4,5	2,3,4,1	2,3,1,4	14,15,13,12
3/15	3,4,5,6	3,4,5,2	3,4,2,5	3,4,2,1

Start \ Pattern	$-++$	$-+-$	$--+$	$---$
1/13	13,12,14,15	13,12,14,11	13,12,11,14	13,12,11,10
2/14	2,1,3,4	14,13,15,12	14,13,12,15	14,13,12,11
3/15	3,2,4,5	3,2,4,1	3,2,1,4	15,14,13,12

TABLE 2

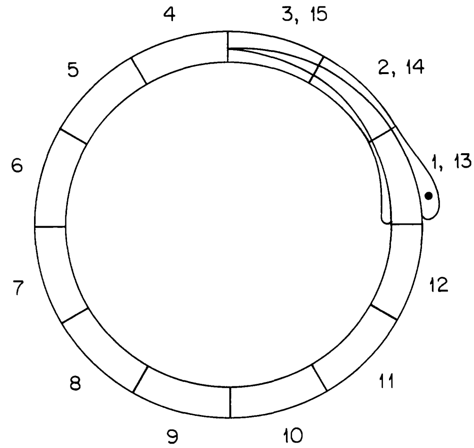

FIGURE 1

Some history

I first heard the original problem in 1952, when I was a graduate student at Harvard, by which time it already seemed to have the status of a "folk theorem." It was used in the 1965 Putnam Competition [**1**], where the published solution uses mathematical induction (yet another approach!), and the comment is made that "Several counting techniques were used by the contestants and many were quite ingenious." (From this it follows that "several $\geqslant$ many.") Equivalent formulations of the problem also appear in books on combinatorial analysis by Liu [**2**] and by Tucker [**3**]. Tucker asks the reader to find a recursion relating the number of solutions for the case $n+1$ to the number of solutions for the case n. The generalization presented above appears to be new, and represents yet another way to solve the original problem.

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[1] 26th William Lowell Putnam Competition, November 20, 1965. See Amer. Math. Monthly, 73 (1966) 728, 730.
[2] C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, 1968, p. 23, exercise 1–25.
[3] Alan Tucker, Applied Combinatorics, John Wiley & Sons, 1980, p. 120, exercise 19, section 4.1.

Comments on the Cover Illustration: Torus with Complete 7-Map

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The cover illustration shows four views of a torus covered by a map of seven hexagonal regions, every two of which are adjacent. Two complementary techniques were used to render these images: ray-casting and parametric patches. The first used the representation of the torus as a quartic surface $T(x, y, z) = 0$. At each point (x, y), or pixel, of the screen a line through the point parallel to the z-axis is intersected with the surface. This yields a quartic equation in z, whose