Taking $n = 1$ leads to two spheres. For $n = 2$ we have a shifted version of the choice $p = cx^2$ treated in our earlier example. More generally, for any n, if k is chosen to make $q \pm p \geq 0$ on an interval I, then $y = \sqrt{q - p}$ and $z = \sqrt{q + p}$ are a pair of solutions to (1) for which the radicands are polynomials. Conversely, all pairs whose squares are polynomials which satisfy (1) are of this form.

There are several possible places in the undergraduate curriculum where our discussion might well be utilized. Whether the lateral area of a cylinder or a similarly dimensioned sphere has a larger area is a provocative way to challenge student intuition when such surfaces are first discussed in a calculus class. The same question could be posed about our ellipsoid-hyperboloid example. A project for a student might be to make various choices for z and then investigate (1) as a nonlinear differential equation in for y. Use of some ODE software might be appropriate, since general solutions are typically unavailable.

References

1. Arthur C. Segal, A Note on the Surface of a Sphere, *The Two-Year College Mathematics Journal* (1982) 63–64.
2. Man-Keung Siu, On the Sphere and Cylinder, *The College Mathematics Journal* (1984) 326–328.

A Tricky Linear Algebra Example

David Sprows (david.sprows@villanova.edu), Villanova University, Villanova, PA

In this note, we develop a result on linear combinations from a vector space that starts out with a little trick. Before the class begins, the instructor writes the number 65 on a piece of paper, then in class, the instructor claims to have the psychic ability to predict sums in advance. The numbers from 1 to 25 are then written consecutively in a 5-by-5 array, as shown below in (A). A student is asked to pick any five numbers from this array with the only restriction being that no two of these numbers can lie in the same row or column. For example, the numbers selected might be the five numbers given in bold face in (B). The student is then asked to add these numbers together and as the student is announcing the result, the instructor shows the class the paper with the number 65 written on it.

$$
\text{(A)}\quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array}
\qquad\qquad
\text{(B)}\quad \begin{array}{ccccc} 1 & \mathbf{2} & 3 & 4 & 5 \\ 6 & 7 & 8 & \mathbf{9} & 10 \\ \mathbf{11} & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & \mathbf{20} \\ 21 & 22 & \mathbf{23} & 24 & 25 \end{array}
$$

Of course, the reason this trick works is that the procedure given above will always lead to the sum of 65. A discussion of this trick leads to some natural questions. For example, does this trick extend to values other than 1 through 25? That is, if we write out the numbers 1 through n^2 consecutively in an n-by-n array, will it always be the case that n numbers selected so that no two are in the same row or column always give the same sum? More generally, how do we determine when an arbitrary n-by-n array of numbers will have this property?

To answer such questions, it is useful to have some terminology. Given an n-by-n matrix A, a set of numbers with one from each row and each column is called a *transversal*. We say that A is a *constant transversal matrix* if all its transversals have the same sum. Note that if A and B have the constant transversal property, then so do $A + B$ and cA for any constant c. It therefore follows that the set of constant transversal matrices forms a vector space V_n.

Clearly any square matrix with constant rows (or with constant columns) has the constant transversal property. We claim that if one row of a constant transversal matrix is constant, then all of the rows are constant. To see why, assume that A is a constant transversal matrix and that one of its rows, say the first, is constant. We will show that it is impossible for any two entries in a given row to be unequal. For suppose that $a_{ij} \neq a_{ik}$ in some row i. Consider any transversal of the $(n-2)$-by-$(n-2)$ submatrix obtained by deleting rows 1 and i and columns j and k from A. If we then extend this transversal to a transversal of A by including a_{1k} and a_{ij}, then the sum of this transversal is some constant c. Next extend the given transversal of the $(n-2)$-by-$(n-2)$ matrix to a transversal of A by including a_{1j} and a_{ik}. Again the sum of this new transversal is the constant c. Since the first row consists of all the same values, we have $a_{1k} = a_{1j}$. It follows that a_{ij} must equal a_{ik}. Clearly the corresponding result holds for columns.

For $k = 1, 2, \ldots, n$, we let R_k denote the n-by-n matrix in which each entry of the kth row is 1 and all other entries are 0, and similarly we let C_k denote the matrix in which each entry of the kth column is 1 and all other entries are 0. Clearly any n-by-n matrix with constant rows is a linear combination of $R_1, R_2, \ldots, R_n$, and likewise for columns. We now proceed to show that $2n-1$ of these matrices span our space V_n.

To this end, let A be a constant transversal matrix, that is, let A be in V_n. As noted earlier, adding or subtracting the same number to every entry in a column does not alter this property. So if for each j we subtract a_{1j} from the jth column of A, then the resulting matrix $B = A - a_{11}C_1 - a_{12}C_2 - \cdots - a_{1n}C_n$ will have the constant transversal property.

Furthermore, since every entry in its first row is 0, all its rows must be constant, with $b_{ij} = b_{i1} = a_{i1} - a_{11}$. It follows that

$$A = (a_{21} - a_{11})R_2 + \cdots + (a_{n1} - a_{11})R_n + a_{11}C_1 + \cdots + a_{1n}C_n. \qquad (*)$$

That is, the set $B_n = \{R_2, R_3, \ldots, R_n, C_1, C_2, \ldots, C_n\}$ spans V_n. (Note that R_1 is not needed; why not?) [We leave it to the reader to show that B_n is a basis for V_n.] Observe that the matrix we started with is $5R_2 + 10R_3 + 15R_4 + 20R_5 + C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5$.

Our result can be used to create other, perhaps more impressive, constant transversal matrices. One way is to start with any $2n-1$ entries as the first row and first column, and then compute A according to our linear combination $(*)$. For example, starting with the array on the left, we get the constant transversal matrix on the right. For this matrix the transversal constant is Hardy's "uninteresting" number 1729.

1	201	302	504	708
2				
3				
5				
7				

1	201	302	504	708
2	202	303	505	709
3	203	304	506	710
5	205	306	508	712
7	207	308	510	714

We now return to our original question concerning an array with the entries $1, 2, \ldots, n^2$. The key to showing that an n-by-n matrix A is a linear combination of $R_2, \ldots, C_n$ is to show that for each value of k we have $a_{ik} - a_{i1} = a_{jk} - a_{j1}$ for all i and j. In particular, if we let A be the n-by-n matrix formed by listing the numbers from 1 to n^2 consecutively, then $a_{ij} = (i-1)n + j$. This means that $a_{ik} - a_{i1} = k - 1$ for every i. Thus we can do the trick described at the start of this note not just for the first 25 numbers, but for the first n^2 numbers for any n, and the predicted sum will always be the sum of the diagonal entries, $\frac{1}{2}(n^3 + n)$.

A Quick Change of Base Algorithm for Fractions

Juan B. Gil (jgil@psu.edu) and Michael D. Weiner (mweiner@psu.edu), Penn State Altoona, Altoona, PA 16601

In this note we discuss the digital (floating-point) representation in various arithmetic bases of a fraction $\frac{1}{m}$ with $m \in \mathbb{N}$. With a base b and a denominator m, we associate a corresponding *key*, defined as the string of digits consisting of the residues of b modulo m, and will use it to speed up some computations. For example, if m is a prime for which $\frac{1}{m}$ has a maximum period expansion of $m-1$ digits, and if b is a primitive root mod m, then the associated key can be used to obtain quickly the digital representation of $\frac{i}{m}$ for $i = 2, 3, \ldots, m-1$, from the representation of $\frac{1}{m}$. Recall that if m is prime, b is a primitive root mod m if $b^j \not\equiv 1 \pmod{m}$ for $1 \le j < m-1$.

On the other hand, for arbitrary integers b and m greater than 1, we will use the key to give a surprisingly simple algorithm to change the representation of $\frac{1}{m}$ in base b to its representation in base $b + mt$ for any $t \in \mathbb{N}$. Our arguments rely on basic modular arithmetic and well-known results that can be found in any textbook on elementary number theory; see, for example, [**1**] or [**2**].

Fractions with cyclic periods. Let us start with the simple and commonly used example of $m = 7$. The number $\frac{1}{7}$ has a couple of fascinating properties that delight even those who are familiar with the mysteries of math. In the decimal system $\frac{1}{7} = 0.\overline{142857}$. Let $\langle 132645 \rangle$ be the key associated with 10 and 7 by means of the residues

$$10^0 \equiv \mathbf{1} \pmod 7, \quad 10^1 \equiv \mathbf{3} \pmod 7, \quad 10^2 \equiv \mathbf{2} \pmod 7,$$
$$10^3 \equiv \mathbf{6} \pmod 7, \quad 10^4 \equiv \mathbf{4} \pmod 7, \quad 10^5 \equiv \mathbf{5} \pmod 7,$$

and let $k(i)$ be the digit in the period 142857 of $\frac{1}{7}$ that corresponds to the digit i in the key

$$\begin{array}{cccccc} 1 & 3 & 2 & 6 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 2 & 8 & 5 & 7 \end{array},$$

that is, $k(1) = 1$, $k(2) = 2$, $k(3) = 4$, and so on. Then $\frac{i}{7} = 0.\overline{k(i)\cdots}$, where the missing 5 digits in the period are placed as to get a rotation of the period of $\frac{1}{7}$. That is,

$$\tfrac{\mathbf{2}}{7} = 0.\overline{\mathbf{2}85714}, \quad \tfrac{\mathbf{3}}{7} = 0.\overline{\mathbf{4}28571}, \quad \tfrac{\mathbf{4}}{7} = 0.\overline{\mathbf{5}71428}, \quad \tfrac{\mathbf{5}}{7} = 0.\overline{\mathbf{7}14285}, \quad \tfrac{\mathbf{6}}{7} = 0.\overline{\mathbf{8}57142}.$$

