Introduction  In a recent article MacKinnon [1] describes four methods that may be used to find square roots of $2 \times 2$ matrices. The first of these methods requires that the matrix for which the square roots are sought be diagonalizable and, subsequently, this method was used by Scott [2] to determine all the square roots of $2 \times 2$ matrices. A surprising conclusion is that scalar $2 \times 2$ matrices possess double-infinities of square roots whereas nonscalar $2 \times 2$ matrices have only a finite number of square roots.

The purpose of this article is to show how the Cayley-Hamilton theorem may be used to determine explicit formulae for all the square roots of $2 \times 2$ matrices. These formulae indicate exactly when a $2 \times 2$ matrix has square roots, and the number of such roots.

By definition, the square roots of a $2 \times 2$ matrix, $A$, are those $2 \times 2$ matrices, $X$, for which

$$X^2 = A. \quad (1)$$

However, for each square matrix $X$, the Cayley-Hamilton theorem states that

$$X^2 - (\text{tr } X)X + (\det X)I = 0. \quad (2)$$

Thus, if a $2 \times 2$ matrix $A$ has a square root $X$, then we may use (2) to eliminate $X^2$ from (1) to obtain

$$(\text{tr } X)X = A + (\det X)I.$$

Further, since $(\det X)^2 = \det X^2 = \det A$, then $\det X = \varepsilon_1 \sqrt{\det A}$, that is $\det \sqrt{A} = \varepsilon_1 \sqrt{\det A}$, so that the above result simplifies to the identity:

$$(\text{tr } X)X = A + \varepsilon_1 \sqrt{\det A} I, \quad \varepsilon_1 = \pm 1. \quad (3)$$

Case 1: A is a scalar matrix. If $A$ is a scalar matrix, $A = aI$, then (3) gives

$$(\text{tr } X)X = (1 + \varepsilon_1)aI, \quad \varepsilon_1 = \pm 1.$$ 

Hence, either $(\text{tr } X)X = 0$ or $(\text{tr } X)X = 2aI$. The first of these possibilities determines the general solution of (1) as

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad \alpha^2 + \beta \gamma = a \quad (4a)$$

and it covers the second possibility if $a = 0$. On the other hand, if $a \neq 0$ then the second possibility, $(\text{tr } X)X = 2aI$, implies $X$ is scalar and has only the pair of solutions

$$X = \pm \sqrt{a} I. \quad (4b)$$

For this case we conclude that if $A$ is a zero matrix then it has a double infinity of square roots as given by (4a) with $a = 0$, whereas if $A$ is a nonzero, scalar matrix then
it has a double-infinity of square roots plus two scalar square roots as given by (4a) and (4b).

Case 2: A is not a scalar matrix. If A is not a scalar matrix then \( \text{tr} \, X \neq 0 \) in (3). Consequently, every square root \( X \) has the form:

\[
X = \tau^{-1} \left( A + \varepsilon_1 \sqrt{\det A} I \right), \quad \tau \neq 0
\]

Substituting this expression for \( X \) into (1) and using the Cayley-Hamilton theorem for \( A \) we find

\[
A^2 + (2\varepsilon_1 \sqrt{\det A} - \tau^2)A + (\text{det} \, A) I = 0
\]

\[
((\text{tr} \, A) A - (\text{det} \, A) I) + (2\varepsilon_1 \sqrt{\det A} - \tau^2)A + (\text{det} \, A) I = 0
\]

\[
(\text{tr} \, A + 2\varepsilon_1 \sqrt{\det A} - \tau^2)A = 0.
\]

Since \( A \) is not a scalar matrix then \( A \) is not a zero matrix, so

\[
\tau^2 = \text{tr} \, A + 2\varepsilon_1 \sqrt{\det A}, \quad (\tau \neq 0, \varepsilon_1 = \pm 1).
\]  

(5)

If \((\text{tr} \, A)^2 \neq 4 \text{det} \, A\) then both values of \( \varepsilon_1 \) may be used in (5) without reducing \( \tau \) to zero. Consequently, it follows from (3) that we may write \( X \), the square root of \( A \), as

\[
X = \varepsilon_2 \frac{A + \varepsilon_1 \sqrt{\det A} I}{\sqrt{\text{tr} \, A + 2\varepsilon_1 \sqrt{\det A}}}.
\]  

(6a)

Here each \( \varepsilon_i = \pm 1 \), and if \( \text{det} \, A \neq 0 \) the result determines exactly four square roots for \( A \). However, if \( \text{det} \, A = 0 \) then result (6a) determines two square roots for \( A \) as given by

\[
X = \pm \frac{1}{\sqrt{\text{tr} \, A}} A.
\]  

(6b)

Alternatively, if \((\text{tr} \, A)^2 = 4 \text{det} \, A \neq 0\), then one value of \( \varepsilon_1 \) in (5) reduces \( \tau \) to zero whereas the other value yields the results: \( 2\varepsilon_1 \sqrt{\det A} = \text{tr} \, A \) and \( \tau^2 = 2 \text{tr} \, A \). In this case, \( A \) has exactly two square roots given by

\[
X = \pm \frac{1}{\sqrt{2 \text{tr} \, A}} \left( A + \frac{1}{2} (\text{tr} \, A) I \right).
\]  

(6c)

Finally, if \((\text{tr} \, A)^2 = 4 \text{det} \, A = 0\) then both values of \( \varepsilon_1 \) reduce \( \tau \) to zero in (5). Hence it follows by contradiction that \( A \) has no square roots.

For this case we conclude that a nonscalar matrix, \( A \), has square roots if, and only if, at least one of the numbers, \( \text{tr} \, A \) and \( \text{det} \, A \), is nonzero. Then the matrix has four square roots given by (6a) if

\[
(\text{tr} \, A)^2 \neq 4 \text{det} \, A, \text{det} \, A \neq 0
\]

and two square roots given by (6b) or (6c) if

\[
(\text{tr} \, A)^2 \neq 4 \text{det} \, A, \text{det} \, A = 0 \quad \text{or} \quad (\text{tr} \, A)^2 = 4 \text{det} \, A, \text{det} \, A \neq 0.
\]

It is worth noting from (6a) that

\[
\text{tr} \, X = \text{tr} \, \sqrt{A} = \varepsilon_2 \sqrt{\text{tr} \, A + 2\varepsilon_1 \sqrt{\det A}}.
\]
Hence using the identity, \( \det \sqrt{A} = e_1 \sqrt{\det A} \) as applied in (3), result (6a) may be rewritten as

\[
\sqrt{A} = \frac{1}{\text{tr}\sqrt{A}} (A + \det \sqrt{A} I),
\]

which is equivalent to the Cayley-Hamilton theorem for the matrix \( \sqrt{A} \). This same deduction can be made, of course, for all other cases under which \( \sqrt{A} \) exists.

REFERENCES


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Proof without Words:
Every Cube Is the Sum of Consecutive Odd Numbers

\[
1^3 = 1 \\
2^3 = 3 + 5 \\
3^3 = 7 + 9 + 11 \\
4^3 = 13 + 15 + 17 + 19 \\
\vdots \\
n^3 = [n(n - 1) + 1] + \cdots + [n(n + 1) - 1]
\]

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