square leads to $2bc$ being square. Without loss of generality, assume (a, b, c) is primitive and parametrize. Then by standard arguments, the parameters, m and n, are both square. Finally, the square $2bc = 4mn(m^2 + n^2)$ implies a solution to the unsolvable $x^4 + y^4 = z^2$. Our original (t, s, x) case can also be proved this way with the argument ending at the unsolvable $x^4 - y^4 = z^2$ instead.

Many authors have considered similar problems. For example, Gerry Myerson noticed the triple $(27^2, t(80), t(81))$. Sierpinski found $(t(132), t(143), t(164))$, the only known (t, t, t) example, as well as the infinite family of triples with consecutive triangular legs like $(t(6), t(7), 35)$. And R. P. Burn found many examples of (t, t, x) which are not included in Sierpinski's work such as $(t(8), t(14), 111)$.

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Bernstein's Examples on Independent Events

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In 1946, S.N. Bernstein [**1**, p. 47] gave two examples showing that if the events in a set are *pairwise* independent, they need not be *jointly* independent. In both examples, the sample space has four outcomes, all equally likely. This raises the question of whether there are smaller examples or others of the same size. In this note, we show that the answer to both parts is that there are not. For the sake of simplicity, all of the sample spaces we discuss are assumed to be finite and to have at least two outcomes, with each outcome having positive probability.

We recall the key definitions. Given an experiment, two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$. More generally, events $A_1, A_2, \ldots, A_k$ are *jointly independent* if $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_j})$ for every subset $\{i_1, i_2, \ldots, i_j\}$ of $\{1, 2, \ldots, k\}$. Obviously, jointly independent sets are pairwise independent.

What Bernstein's examples show is that the converse is not true. For our version of his first example [**1**, p. 47], we consider an urn containing four balls, numbered 110, 101, 011 and 000, from which one ball is drawn at random. For $i = 1, 2, 3$ let A_i be the event of drawing a ball with 1 in the ith position. Thus, the three events are pairwise independent. However, since $A_1 \cap A_2 \cap A_3 = \varnothing$, they are not jointly independent.

In his second example, Bernstein used a tetrahedron with colored faces (one red, one blue, one green, and one with all three colors). We give an equivalent example using the same sample space as in the first example, but with events A_1, A_2, and A_3 where A_i is the event of drawing a ball with a 0 in the ith position, not a 1. Note that each $P(A_i) = \frac{1}{2}$ and each $P(A_i \cap A_j) = \frac{1}{4}$ for $i \neq j$, so the three events are again pairwise independent. Now however, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{4}$, while $P(A_1)P(A_2)P(A_3) = \frac{1}{8}$. Thus, we again have three pairwise independent events that are not jointly independent.

We note that the two examples are intrinsically different since in one case the intersection of the three events is empty while in the other it is not. This difference can also be seen in the diagrams in Figure 1.

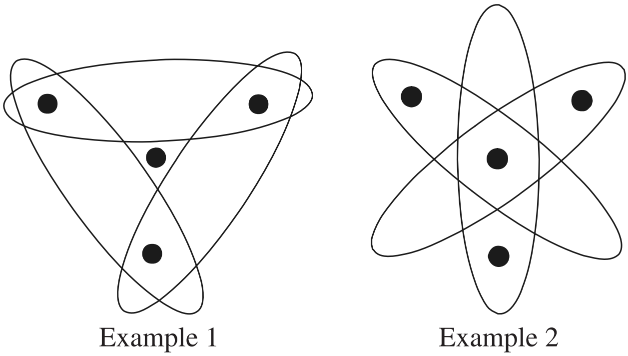

Figure 1. Configurations of events.

We now show that these are essentially the only examples with no more than four outcomes in the sample space. We assume that we have an experiment on a set of n outcomes with $n \leq 4$ and that $A_1, A_2, \ldots, A_k$ $(k \geq 3)$ is a collection of dependent events that are pairwise independent. We will show that $n = 4$, $k = 3$, and that the sets form one of the arrangements in Figure 1. We also show that the space is homogeneous (all of the outcomes have the same probability).

Note that we may (and do) assume that none of our events is the whole sample space since its presence does not affect either the overall dependence or the pairwise independence. We next suppose that one, say A_1, consists of a single element x, and consider A_2. If $x \notin A_2$, then A_1 and A_2 are disjoint and thus not independent, while if $x \in A_2$, then $P(A_1 \cap A_2) = P(A_2)$, so again A_1 and A_2 are not independent. Hence, each A_i must have at least two elements. It is a well-known fact that if two events are independent so are their complements, so just as no A_i can have just one element, neither can its complement. Therefore each A_i must have exactly two elements, and n must be 4.

There are now two cases to consider:

Case 1. No outcome is in all three events. Then we may assume that $A_1 = \{w, x\}$, $A_2 = \{w, y\}$, and $A_3 = \{x, y\}$, and so we have Configuration 1.

Case 2. Some outcome is in all three events. Then we may assume that $A_1 = \{w, x\}$, $A_2 = \{w, y\}$, and $A_3 = \{w, z\}$, and so we have Configuration 2.

All that remains is to show that the space is homogeneous. We do this for Case 2, leaving Case 1 to the reader. Let p, q, r, and s denote the probabilities of w, x, y, and z, respectively. It follows from the pairwise independence of the three events that $(p + q)(p + r) = p$, $(p + q)(p + s) = p$, and $(p + r)(p + s) = p$. Since $p + q + r + s = 1$, a bit of algebra yields $p = q = r = s = \frac{1}{4}$.

Thus, Bernstein's examples are essentially the only ones with four outcomes in the sample space, and there are none with fewer.

Acknowledgment. The author wishes to express his appreciation to the editors for their valuable suggestions.

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An Improper Application of Green's Theorem

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The improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx \tag{1}$$

converges to $\pi/2$, but how is this value calculated? Since the antiderivative of $f(x) = \sin x/x$ cannot be expressed as a finite combination of elementary functions (see [**1**]), we must look beyond the Fundamental Theorem of Calculus. We present here a way to calculate (1) using only techniques covered in calculus, but we first present two standard calculations.

First, consider the Laplace transform F of $\sin x/x$:

$$F(s) = \int_0^\infty \frac{\sin x}{x} e^{-sx}\,dx.$$

As s becomes large, the kernel e^{-sx} converges rapidly to 0 as long as $x > 0$. Since $|\sin x/x| < 1$ for $x > 0$, we have $|F(s)| < 1/s$, which gives $\lim_{s\to\infty} F(s) = 0$.

Assuming that differentiation with respect to s may be perfomed either before or after the integration, we have

$$\frac{dF}{ds} = -\int_0^\infty (\sin x)e^{-sx}\,dx = \frac{-1}{s^2+1}.$$

Hence $F(s) = -\arctan(s) + C$, where C is a constant. But $\lim_{s\to\infty} F(s) = 0$, so C must be $\pi/2$. Thus, since $C = F(0)$,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2.$$

For students with a solid background in integral calculus, the broad strokes of this calculation are easy to follow. We took a leap, though, when we differentiated with respect to s. This step, while justifiable, goes beyond topics covered in most calculus texts (conditions under which derivatives may be passed across integral signs are not given in [**3**], for example).

A second calculation of (1) is found in [**4**] (see pp. 278 and 189, Problems 6.14 and 6.15). This calculation starts with an improper *double* integral over the first quadrant: