On the Rational Solutions of $x^y = y^x$

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The number 16 has the interesting property that it can be written as a power of a positive integer in two ways:

$$16 = 2^4 = 4^2.$$

The question arises naturally: Are there any other pairs of integers (x, y) such that

$$x^y = y^x? \tag{1}$$

The problem is not new and not too difficult, but it does not appear in standard texts, so it is not widely known.

In a letter by Daniel Bernoulli to Goldbach (1728) [1], equation (1) is mentioned with the statement that $(x, y) = (2,4)$ (or $(4,2)$) is the only integer solution. In his answer Goldbach gives the general solution of (1) by writing

$$y = ax,$$

hence, $x^{ax} = (ax)^x$ and after simplification, and ignoring the trivial case when $a = 1$,

$$x = a^{1/(a-1)} \quad \text{and} \quad y = a^{a/(a-1)}. \tag{2}$$

Equation (1) is also discussed in some detail by Euler in [**2**].

The graph of the implicit function $y^x - x^y = 0, (x > 0, y > 0)$ is given as shown in FIGURE 1.

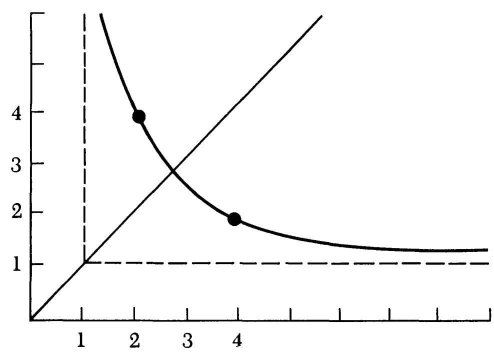

FIGURE 1

Setting $u = 1/(a-1)$ in (2), obtain

$$x = \left(1 + \frac{1}{u}\right)^u \quad \text{and} \quad y = \left(1 + \frac{1}{u}\right)^{u+1}. \tag{3}$$

Thus the graph consists of two branches: The case $a = 1$ yields the line $x = y$, while the parametric equation (3) is represented by the curve symmetrical about the above line and having $x = 1$ and $y = 1$ as asymptotes (as $u \to 0^+$ and $u \to 0^-$, respectively). The intersection of the curve and the line represents

$$\left(\lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^{u}, \lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^{u+1} \right) = (e, e).$$

The only lattice points on the curved branch are (2,4) and (4,2), but setting integer values for u in (3), Euler obtains an infinite number of isolated points with rational coordinates:

$$\left(\frac{9}{4}, \frac{27}{8}\right), \left(\frac{64}{27}, \frac{256}{81}\right) \dots$$

It seems that around the turn of the century equation (1) was quite fashionable. L. E. Dickson, in his *History of the Theory of Numbers* [3], cites a number of contributions prior to 1951. Noteworthy and easily accessible is a discussion of the complete locus for real x and y by E. J. Moulton [4].

The problem surfaced again in 1960 as a Putnam Competition question asking for the integer solutions of (1). This prompted A. Hausner to extend results to algebraic number fields [5].

The approach to be described in the present note is aimed at finding *all the nontrivial rational solutions* of (1). It will be shown that these are given by a sequence

$$(a_n, b_n) \to (e, e),$$

where a_n and b_n turn out to be identical to the expressions (3) found by Goldbach and Euler, where positive integers are substituted into (3).

Discarding the trivial solution $x = y$ of (1), we assume that $y > x$, and write (1) in the form

$$y = x^{y/x},$$

or dividing by x, obtain

$$x^{(y/x)-1} = \frac{y}{x}. \tag{4}$$

Set

$$\frac{y}{x} - 1 = \frac{m}{n}, \tag{5}$$

where m, n are positive integers. Furthermore we assume that m/n is reduced, hence the greatest common divisor

$$(m, n) = 1.$$

Equation (4) becomes

$$x^{m/n} = \frac{m+n}{n}$$

or

$$x = (m+n)^{n/m}/n^{n/m}. \tag{6}$$

Since m and n are coprime, so are $m+n$ and n, hence also

$$((m+n)^n, n^n) = 1.$$

It follows then from (6) that x is rational if and only if both $(m+n)^n$ and n^n are mth powers.

This implies that each of $m+n$ and n must be an mth power, since m and n (treated now as exponents) are coprime.

Hence

$$n = a^m$$

and

$$m + n = b^m,$$

where a, b are positive integers and $b > a$.

This is *possible if and only if* $m = 1$, since the difference between two consecutive mth powers is $> m$, if $m > 1$.

So by (6)

$$x = \left(1 + \frac{1}{n}\right)^n$$

and from (4)

$$y = \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for } n = 1, 2, \ldots .$$

In particular, *the only integer solution* of (1) when $y > x$, is obtained when $n = 1$, giving the result

$$x = 2, \qquad y = 4.$$

It is interesting to look graphically at the above results. We want pairs (x, y) for which

$$x^{1/x} = y^{1/y} \tag{7}$$

or, equivalently,

$$\frac{1}{x}\ln x = \frac{1}{y}\ln y. \tag{8}$$

From the graph of $f(t) = \frac{1}{t}\ln t$ $(t > 0)$ (FIGURE 2) it is easy to see that for all $t > e$ there is exactly one value t', where $1 < t' < e$ such that $f(t') = f(t)$.

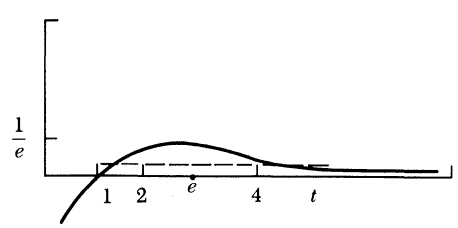

FIGURE 2

Thus a bijective map exists between the *points* on the graph for $t > e$ and pairs (x, y) satisfying (7). The points corresponding *to rational pairs* are *all* on the curve *over the interval* $e < t \leqslant 4$, an illustration of the "scarcity" of rational solutions.

The author thanks the referee for drawing her attention to the history of the problem.

REFERENCES

1. *Correspondence Math. Phys.* (edited by Fuss), Vol. 2, pp. 262 and 280.
2. Leonhard Euler, *Introductio in analysin infinitorum*, Tom II, Cap. 21 §519.
3. L. E. Dickson, *History of the Theory of Numbers*, New York, Vol. II, Geneva and Lausanne, 1748, p. 687.
4. E. J. Moulton, The real function defined by $X^y = Y^x$, *Amer. Math. Monthly* 23 (1916), 233–237.
5. A. Hausner, Algebraic number fields and the Diophantine equation $m^n = n^m$, *Amer. Math. Monthly* 68 (1961), 856–861.

Wronskian Harmony

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Recently J. Wichmann in Osnabrück considered the Wronski-type determinants for the harmonic oscillators $\sin x, \sin 2x, \ldots, \sin kx$. He found these determinants for $k \leq 7$ to be a coefficient times a power of $\sin x$, and hence suggested the formula

$$f_n(x) = \det\left(\frac{d^j(\sin(k+1)x)}{dx^j}\right)_{j,k=0,\ldots,n-1} = K_n(\sin x)^{S_n}, \tag{1}$$

with $S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and the coefficients K_n to be explored, e.g.

$$\begin{aligned}
f_1(x) &= \sin x\\
f_2(x) &= -2\sin^3 x\\
f_3(x) &= -16\sin^6 x\\
f_4(x) &= 768\sin^{10} x\\
f_5(x) &= 294912\sin^{15} x\\
f_6(x) &= -1132462080\sin^{21} x\\
f_7(x) &= -52183852646400\sin^{28} x\\
&\text{etc.}
\end{aligned}$$

The elusive formula cannot be completely trivial since it must depend on properties which the sine does not share with the cosine. The corresponding determinants with "cos" substituted for "sin" give several terms, e.g.,

$$\begin{vmatrix} \cos x & \cos 2x \\ -\sin x & -2\sin 2x \end{vmatrix} = 2\sin^3 x - 3\sin x$$

$$\begin{vmatrix} \cos x & \cos 2x & \cos 3x \\ -\sin x & -2\sin 2x & -3\sin 3x \\ -\cos x & -4\cos 2x & -9\cos 3x \end{vmatrix} = 16\cos x\sin^5 x - 40\cos x\sin^3 x.$$

On the other hand, if we substitute e^x for $\sin x$, then we obtain a formula similar to