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More on the Lost Cousin of the
Fundamental Theorem of Algebra

ROMAN SZNAJDER
Bowie State University
Bowie, MD 20715-9465
rsznajder@bowiestate.edu

In his recent note [2], Timo Tossavainen proves what he calls “The Lost Cousin of the Fundamental Theorem of Algebra,” which we state as:

**Exponential Theorem.** For any integer \( n \geq 1 \), let \( 0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n \) and \( a_j \) (for \( j = 0, \ldots, n \)) be real numbers with \( a_n \neq 0 \). Then the function \( f : \mathbb{R} \to \mathbb{R}, \)

\[
f(t) = \sum_{j=0}^{n} a_j \kappa_j^j
\]

has at most \( n \) zeros.

Years ago, I was presented by a friend with a copy of a concise monograph [1] (112 pages long) on selected topics in polynomial approximation. In this book, apparently unknown to western readers, the following fact and its proof appear:

**Generalized Polynomial Theorem.** A function \( g \) given by the formula

\[
g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n},
\]

where \( \alpha_0 < \alpha_1 < \cdots < \alpha_n \) are arbitrary real numbers and \( a_n \neq 0 \), has no more than \( n \) roots.

**Proof.** We proceed by induction on \( n \), noting that for \( n = 1 \) the statement is obvious. Assume that for some \( n \) the claim is true, but for \( n + 1 \), it is not. Hence, for some real numbers \( \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} \) and \( a_{n+1} \neq 0 \), there is a function

\[
g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n} + a_{n+1} x^{\alpha_{n+1}},
\]

whose number of positive roots is larger than \( n + 1 \). These roots are identical with the roots of the new function

\[
g(x)/x^{\alpha_0} = a_0 + a_1 x^{\alpha_1 - \alpha_0} + \cdots + a_n x^{\alpha_n - \alpha_0} + a_{n+1} x^{\alpha_{n+1} - \alpha_0}.
\]

By Rolle’s theorem, the derivative of the above function, which has the form

\[
b_0 x^{\beta_0} + b_1 x^{\beta_1} + \cdots + b_n x^{\beta_n},
\]
with $\beta_0 < \beta_1 < \cdots < \beta_n$ and $b_n \neq 0$, has more than $n$ roots. This contradiction to the induction hypothesis concludes the proof.

The Exponential Theorem generalizes the fundamental theorem of algebra to exponential functions the way the Generalized Polynomial Theorem does for generalized polynomials. A striking fact is that the two proofs follow the same path. Despite appearances, the theorems are equivalent, as the following argument shows.

Let $f(t) = \sum_{j=0}^{n} a_j \kappa_j t^j$ with $0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$, $a_j \in \mathbb{R}$, and $a_n \neq 0$. Let $\kappa_0 = e^{c_0}, \kappa_1 = e^{c_1}, \ldots, \kappa_n = e^{c_n}$ for some $c_0 < c_1 < \cdots < c_n$. By multiplying $f(t)$ by $\Delta^t$ for a suitable $\Delta > 1$, we may assume that $c_0 > 0$ to ascertain that $1 < c_1/c_0 < \cdots < c_n/c_0$. Then

$$f(t) = a_0 e^{c_0 t} + a_1 e^{c_1 t} + \cdots + a_n e^{c_n t}$$

$$= a_0 e^{c_0 t} + a_1 \left( e^{c_0 t} \right)^{c_1/c_0} + \cdots + a_n \left( e^{c_0 t} \right)^{c_n/c_0}$$

$$= a_0 x + a_1 x^{c_1/c_0} + \cdots + a_n x^{c_n/c_0} = g(x)$$

with $x = e^{c_0 t}$. By the Generalized Polynomial Theorem, with $\alpha_0 = 1, \alpha_1 = c_1/c_0, \ldots, \alpha_n = c_n/c_0$, there exist at most $k$ positive roots of the corresponding function $g(x)$. Certainly, when $x_i$ is such a root, $t_i := (\ln x_i)/c_0$ becomes a root of $f(t)$, and vice versa. This way, we have shown that the Generalized Polynomial Theorem implies the Exponential Theorem. The opposite implication comes from reversing the argument.

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Closed Knight’s Tours with Minimal Square Removal for All Rectangular Boards

JOE DEMAIO  
Kennesaw State University  
Kennesaw, GA 30144  
jdemaio@kennesaw.edu

THOMAS HIPPCHEN  
Kennesaw State University  
Kennesaw, GA 30144  
thippchen@gmail.com

Finding a closed knight’s tour of a chessboard is a classic problem: Can a knight use legal moves to visit every square on the board and return to its starting position? [1, 3] An open knight’s tour is a knight’s tour of every square that does not return to its starting position. While originally studied for the standard $8 \times 8$ board, the problem is easily generalized to other rectangular boards. In 1991 Schwenk classified all rectangular boards that admit a closed knight’s tour [2]. He described every board that cannot admit a closed knight’s tour and constructed closed knight’s tours for all other boards.