Since $k/\sqrt{n}$ is almost always very near to $\frac{1}{2}m/\sqrt{n}$, the probability of winning when $m(p)=m$ is very nearly equal to

$$\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{m/2\sqrt{n}} e^{-u^2}\,du.$$

Multiplying this by the probability that $m(p)=m$ as computed above, we find finally

$$\begin{aligned}\text{probability of winning} &\approx \frac{1}{2}+\sum_{m=0}^{\infty}\frac{m}{2n}e^{-m^2/4n}\left(\frac{1}{\sqrt{\pi}}\int_0^{m/2\sqrt{n}} e^{-u^2}\,du\right)\\ &\approx \frac{1}{2}+\int_0^{\infty}\frac{x}{2n}e^{-x^2/4n}\left(\frac{1}{\sqrt{\pi}}\int_0^{x/2\sqrt{n}} e^{-u^2}\,du\right)dx\\ &= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} xe^{-x^2/4}\left(\int_0^{x/2} e^{-u^2}\,du\right)dx\\ &= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\left(\int_{2u}^{\infty} xe^{-x^2/4}\,dx\right)du\\ &= \frac{1}{2}+\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-u^2}(2e^{-u^2})\,du\\ &= \frac{1}{2}+\frac{1}{2\sqrt{2}},\end{aligned}$$

as claimed. This is very nearly 6/7, so the result of our paper can be conveniently implemented by beating one's kids on weekdays and Saturdays, but never on Sunday.

REFERENCE

1. Kenneth M. Levasseur, How to Beat Your Kids at Their Own Game, this MAGAZINE 61 (1988), 301–305.

A Note on the Five-Circle Theorem

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In his paper [1] H. Demir stated and proved

THE FIVE-CIRCLE THEOREM. *Let P and Q be two points on the side BC of a triangle ABC in the order B, P, Q, C. If the triangles ABP, APQ, AQC have congruent incircles, then the triangles ABQ, APC have congruent incircles.*

He also asked for a geometric proof of this theorem.

Here we give such a proof for the following more general

FOUR-CIRCLE THEOREM. *Let P and Q be two points on the side BC of a triangle ABC. Then the triangles ABP and AQC have congruent incircles if and only if the triangles ABQ and APC have congruent incircles.*

Proof. We omit the trivial case when P coincides with Q. Without loss of generality we may assume that P lies between B and Q (see FIGURE 1). Denote by r, r_1, r_2, ρ_1 and ρ_2 the radii of the incircles k, k_1, k_2, k' and k'', respectively, of the triangles APQ, ABP, AQC, ABQ and APC. Let T, T_1 and T_q be the tangency points of the line BC, respectively, with k, k_1 and the excircle k_q of the triangle APQ (see FIGURE 2).

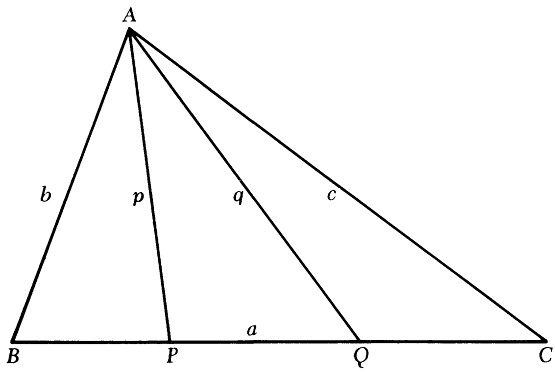

FIGURE 1

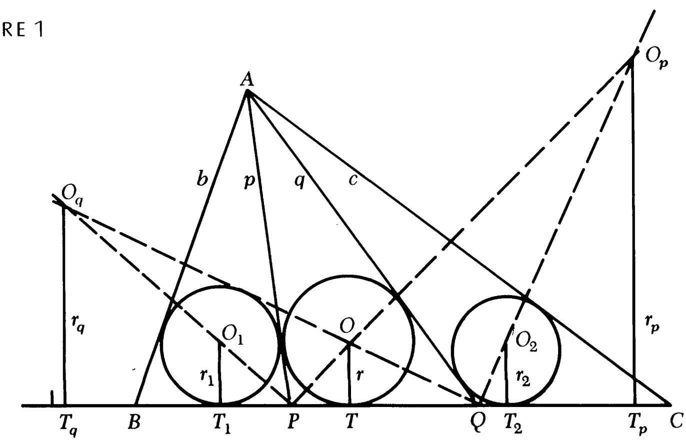

FIGURE 2

From similar triangles, we obtain

$$\frac{r}{r_q} = \frac{QT}{QT_q} \quad \text{and} \quad \frac{r_1}{r_q} = \frac{PT_1}{PT_q},$$

where r_q is the radius of k_q. Therefore,

$$\frac{r}{r_1} = \frac{QT}{QT_q} \cdot \frac{PT_q}{PT_1}. \tag{1}$$

For convenience denote (see FIGURE 1) AB, AP, AQ, AC and PQ by b, p, q, c and a, and the semiperimeters of triangles ABP, APQ and AQC, respectively, by s_1, s and s_2. Then (see [2, p. 87]) $QT = s - p$, $QT_q = s$, $PT_q = QT_q - PQ = s - a$ and $PT_1 = s_1 - b$, and (1) becomes

$$\frac{r}{r_1} = \frac{s-p}{s} \cdot \frac{s-a}{s_1 - b}.$$

Similarly, we obtain

$$\frac{r}{r_2} = \frac{s-q}{s} \cdot \frac{s-a}{s_2 - c}.$$

Hence r_1 and r_2 are equal if and only if

$$(s-p)(s_2 - c) = (s-q)(s_1 - b). \tag{2}$$

In a similar way, let T' and T'' denote the tangency points of BC with k' and k'' (see FIGURE 3). From similar triangles, we obtain

$$\frac{\rho_1}{r} = \frac{QT'}{QT} \quad \text{and} \quad \frac{\rho_2}{r} = \frac{PT''}{PT};$$

therefore, $\rho_1 = \rho_2$ if and only if $PT \cdot QT' = QT \cdot PT''$, that is, if and only if

$$(s-q)(\sigma_1 - b) = (s-p)(\sigma_2 - c), \tag{3}$$

where σ_1 and σ_2 are the semiperimeters of triangles ABQ and APC, respectively.

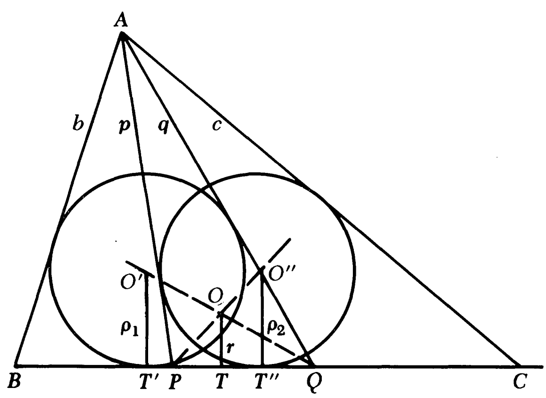

FIGURE 3

But, clearly, $\sigma_1 = s + s_1 - p$ and $\sigma_2 = s + s_2 - q$; hence

$$\begin{aligned}
&(s-q)(\sigma_1 - b) - (s-p)(\sigma_2 - c) \\
&\quad = (s-q)(s + s_1 - p - b) - (s-p)(s + s_2 - q - c) \\
&\quad = (s-q)(s-p) + (s-q)(s_1 - b) - (s-p)(s-q) - (s-p)(s_2 - c) \\
&\quad = (s-q)(s_1 - b) - (s-p)(s_2 - c).
\end{aligned}$$

Consequently (2) and (3) are equivalent, which proves the theorem.

The Four-Circle Theorem can be easily proved also from Demir's equation (4) in the article [1], where this equation is obtained by means of trigonometry.

REFERENCES

1. H. Demir, Incircles within, this MAGAZINE 59 (1986), 77–83.
2. N. Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1968.