Like Pascal’s triangle, Faulhaber’s triangle is easy to draw: all you need is pen, paper and a little recursion. The rows of Faulhaber’s triangle are the coefficients of polynomials that represent sums of integer powers. Such polynomials are often called \textit{Faulhaber formulae} \cite{2}, after Johann Faulhaber (1580–1635); hence we dub the triangle Faulhaber’s triangle.

\section*{Constructing Faulhaber’s triangle}

Draw a right triangle, similar to the one shown in Figure 1. Number the rows, starting with row 0; number the columns from left to right, starting with column 1. The numbers on row \( i \) are found using the following recursive rules:

\begin{itemize}
  \item The leftmost element of each row is chosen such that the row sums to 1. In particular, the only number on row 0 is 1.
  \item The element at row \( i \) and column \( j \) (\( 1 < j \leq i + 1 \)) is found by multiplying the number directly above and to the left by \( \frac{i}{j} \).
\end{itemize}

\begin{verbatim}
row 0         1
row 1       \( \frac{1}{2} \)  \( \frac{1}{2} \)
row 2       \( \frac{1}{6} \)  \( \frac{1}{2} \)  \( \frac{1}{3} \)
row 3       0         \( \frac{1}{4} \) \( \frac{1}{2} \)  \( \frac{1}{4} \)
row 4     \( -\frac{1}{30} \)  0     \( \frac{1}{5} \) \( \frac{1}{2} \)  \( \frac{1}{5} \)
row 5     0     \( -\frac{1}{12} \)  0     \( \frac{5}{12} \) \( \frac{1}{2} \)  \( \frac{1}{6} \)
row 6 \( \frac{1}{42} \)  0     \( -\frac{1}{6} \)  0     \( \frac{1}{2} \) \( \frac{1}{2} \)  \( \frac{1}{7} \)
\ldots
\end{verbatim}

\textbf{Figure 1.} Faulhaber’s triangle

\begin{tiny}
doi:10.4169/college.math.j.42.2.096
\end{tiny}
Sums of integer powers

The sum of integer powers \( 1^p + 2^p + \cdots + n^p \), with integers \( n, p \geq 0 \), is a polynomial in \( n \) of degree \( p + 1 \). That is \( f_p(n) = a_{p+1}n^{p+1} + a_p n^p + \cdots + a_1 n + a_0 \). Taking \( n = 0 \), it follows immediately that \( a_0 = 0 \). In order to find the coefficients of the polynomial, we draw Faulhaber’s triangle. Row \( p \) of the triangle gives the coefficients \( a_1, \ldots, a_{p+1} \).

For instance, to find \( f_4(n) \) we use row 4 of Figure 1: \( a_1 = -\frac{1}{30}, a_2 = 0, a_3 = \frac{1}{3}, a_4 = \frac{1}{2} \) and \( a_5 = \frac{1}{3} \). That is

\[
f_4(n) = \sum_{i=1}^{n} i^4 = 1 \cdot n^5 + 2 \cdot n^4 + 3 \cdot n^3 - \frac{1}{30} n.
\]

We now observe that \( f_p(n) \) is always of the shape

\[
\frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + a_{p-1} n^{p-1} + a_{p-3} n^{p-3} + \cdots,
\]

with all coefficients \( a_{p-2k} = 0 \) for \( k > 0 \). We also note that the numbers appearing on the vertical leg (leftmost column) of Faulhaber’s triangle are the Bernoulli numbers, namely \( B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30} \), etc. This is due to the well-known Bernoulli formula stating

\[
f_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} \binom{p+1}{i} B_i n^{p+1-i}.
\]

Why it works

Suppose the coefficient of \( n^a \) is \( \alpha \) in \( f_b(n) \), for some \( 1 < a \leq b + 1 \), and the coefficient of \( n^{a-1} \) in \( f_{b-1}(n) \) is \( \beta \). It can be shown that \( \alpha = \frac{b}{a} \beta \), cf. [1, 3]. In Faulhaber’s triangle, this corresponds to row \( b - 1 \) containing \( \beta \) at column \( a - 1 \), and row \( b \) containing \( \alpha \) at column \( a \). Note that our construction of Faulhaber’s triangle ensures \( \alpha = \frac{b}{a} \beta \).

Next, observe that \( f_p(1) = a_{p+1} + \cdots + a_1 = 1 \), for all \( p \), so that \( a_1 = 1 - (a_{p+1} + \cdots + a_2) \). This is the reason the leftmost element of each row is chosen such that the values on the row sum up to 1.

Now, by a straightforward induction, if the numbers on row \( p \) are the coefficients of \( f_p(n) \), then the numbers on row \( p + 1 \) are the coefficients of \( f_{p+1}(n) \). The base case is immediate, as \( f_0(n) = n \).

Summary. Like Pascal’s triangle, Faulhaber’s triangle is easy to draw: all you need is a little recursion. The rows are the coefficients of polynomials representing sums of integer powers. Such polynomials are often called Faulhaber formulae, after Johann Faulhaber (1580–1635); hence we dub the triangle Faulhaber’s triangle.

References