Suppose we have a pentagram in the $xy$-plane, oriented as in Figure 1a, and want to find a quartic polynomial whose graph passes through the three vertices indicated. Out of infinitely many possibilities, there is exactly one quartic polynomial that attains its minimum value at both of the two lower vertices. This graph—shaped like a smooth W with its local maximum at the upper vertex—is shown in Figure 1b. Now, how does the graph continue? Will it touch the pentagram again on its way up to infinity? As it turns out, the graph passes through two more vertices, as shown in Figure 1c. Furthermore, the two points where the graph crosses the interior of a pentagram edge lie exactly below two other vertices, as shown in Figure 1d.

Knowing that length ratios within the pentagram are determined by the golden ratio, we realize that this quartic polynomial has some regularities governed by the same ratio. As we will see, there are many more such regularities. Furthermore, they apply to all quartic polynomials with inflection points. This will be clear once we realize that the different quartics are all related by an affine transformation.

**Symmetric quartic** We investigate graphs of quartic polynomials with inflection points by means of certain naturally defined points and length ratios. As an example, we consider the function $f(x) = x^4 - 2x^2$, shown in Figure 2. (This quartic’s shape differs slightly from the one in Figure 1 and is chosen to simplify calculations.) We define $P_0(x_0, y_0)$ as the point where the third derivative vanishes, so that $f'''(x_0) = 0$. The tangent points of the double tangent (the unique line that is tangent to the graph at two points) are called $P_1$ and $P_2$. The points where the tangent at $P_0$ intersects the graph are $P_3$ and $P_4$. We number points so that those to the left of $P_0$ have odd index, while those to the right have even index.

The line through $P_0$ and $P_1$ intersects the graph in two additional points, called $P_6$ and $P_7$. Similarly, the line through $P_0$ and $P_2$ has the additional intersection points $P_5$ and $P_8$. 

**Figure 1** A pentagram and a quartic polynomial
and $P_8$. (The inflection points might appear to be $P_7$ and $P_8$, but this is not so.) The line through $P_7$ and $P_8$ intersects the graph in $P_9$ and $P_{10}$.

For this graph, we easily find the coordinates $P_0(0,0)$, $P_2(1,-1)$, and $P_4(√2,0)$. Symmetry guarantees that the points $P_1$ through $P_{10}$ are located symmetrically about the $y$-axis. The coordinates of $P_5$ through $P_{10}$ turn out to involve the golden ratio, $ϕ = (1 + √5)/2$. Using the relations $ϕ^2 = ϕ + 1$ and $ϕ^{-2} = 1 - ϕ^{-1}$, we calculate three function values: $f(ϕ) = ϕ$ and $f(ϕ^{-1}) = f(√ϕ) = -ϕ^{-1}$. From these calculations and the fact that the points $P_5$ through $P_8$ lie on the lines $y = ±x$, we find the coordinates $P_6(ϕ, ϕ)$, $P_8(1/ϕ, -1/ϕ)$, and $P_{10}(√ϕ, -1/ϕ)$. From these coordinates, the following relations between line segment lengths follow quickly:

$$P_3P_4 = √2P_1P_2,$$

$$P_5P_6 = ϕP_1P_2,$$

$$P_7P_8 = P_1P_2/ϕ,$$

$$P_9P_{10} = √ϕP_1P_2.$$  

(1)

Our next step is to show that these relations carry over to the general case.

**General quartic** We will not proceed by deriving general expressions for the coordinates of the points $P_0$ through $P_{10}$. Instead, we shall see that the graph of every quartic polynomial with inflection points can be obtained as the image of the graph of the symmetric quartic above subject to an appropriate affine transformation. An affine transformation consists of an invertible linear transformation followed by translation along a constant vector. An affine transformation of the plane has the following properties: Straight lines are mapped to straight lines, parallel lines to parallel lines, and tangents to tangents, while length ratios between parallel line segments are preserved [1, chapter 2].

Consider the symmetric quartic

$$f(x) = x^4 + wx^2, \quad w < 0,$$  

(2)

and a general quartic with inflection points,

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a ≠ 0.$$  

Define $x_0$ by $g''(x_0) = 0$ (so $x_0 = -b/4a$) and $k = √g''(x_0)/2aw$. (The existence of two inflection points implies that $g''(x_0)$ and $a$ have opposite signs, and since $w$ is
negative, $k$ is real.) The map $(x, y) \mapsto (\bar{x}, \bar{y})$ given by

\[
\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
g'(x_0) & 1
\end{pmatrix} \begin{pmatrix}
k & 0 \\
0 & ak^4
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
x_0 \\
g(x_0)
\end{pmatrix}
\]

is an affine transformation consisting of a scaling transformation (with a scaling factor $k$ in the $x$-direction and a scaling factor $|a|k^4$ in the $y$-direction), a reflection about the $x$-axis if $a$ is negative, a shear in the $y$-direction, and a translation. In components, we have the equations

\[
\bar{x} = kx + x_0, \quad \bar{y} = ak^4y + g'(x_0)kx + g(x_0).
\]

Suppose $(x, y)$ lies on the graph of $f$, that is, $y = f(x)$. Substituting $y = x^4 + wx^2$ and $x = (\bar{x} - x_0)/k$ into the expression for $\bar{y}$ yields

\[
\bar{y} = a(\bar{x} - x_0)^4 + \frac{1}{2}g''(x_0)(\bar{x} - x_0)^2 + g'(x_0)(\bar{x} - x_0) + g(x_0).
\]

The right-hand side is the fourth-degree Taylor polynomial of $g$ at $x_0$ and is therefore identical to $g(\bar{x})$. Thus, $(\bar{x}, \bar{y})$ lies on the graph of $g$, so the above transformation indeed maps the graph of $f$ to the graph of $g$.

Moreover, the origin, where $f'''(x) = 0$, is mapped to $(x_0, g(x_0))$, where $g'''(x) = 0$. From this and the general properties of affine maps it follows that each of the points $P_0$ through $P_{10}$ on the graph of $f$ is mapped to the analogously defined point on the graph of $g$. (We will use the same notation $P_i$ for points on both graphs.) The results for the case $w = -2$ then show that the line segments $P_1P_2$, $P_3P_4$, $\ldots$, $P_9P_{10}$ on the graph of $g$ are all parallel, are bisected by the vertical line through $P_0$, and satisfy the relations (1). FIGURE 3 illustrates this for the quartic $2x^4 - x^3 - 2x^2 + x + 1$. Note that $P_0$ divides the line segments $P_6P_1$ and $P_5P_2$ according to the golden ratio, $P_7$ divides $P_0P_1$ according to the golden ratio, and analogously for $P_8$ and $P_0P_2$.

**Further characteristic ratios** We now define some more points on the graph of a quartic, starting with the point $P_0$ from FIGURE 3 and the inflection points $P_{11}$ and $P_{12}$;
Figure 4  The quartic $2x^4 - x^3 - 2x^2 + x + 1$ and the points $P_{11}$ through $P_{20}$

see Figure 4. The tangents at the inflection points intersect the graph at the points $P_{13}$ and $P_{14}$. The line through the inflection points intersects the graph at $P_{15}$ and $P_{16}$. The line through $P_0$ and $P_{15}$ intersects the graph at $P_{18}$ and $P_{19}$, while the line through $P_0$ and $P_{16}$ intersects the graph at $P_{17}$ and $P_{20}$. The list of statements may now be extended:

**Theorem.** Let $P_0, \ldots, P_{20}$ be points defined as above on the graph of a quartic polynomial with inflection points, and $\varphi = (\sqrt{5} + 1)/2$. Then:

1. The line segments $P_{2n-1}P_{2n}$ ($n = 1, \ldots, 10$) are all parallel.
2. Intersection points of the graph and a line parallel to the tangent in $P_0$ are symmetrically located about the point on the line with $x = x_0$.
3. $P_3P_4 = \sqrt{2}P_1P_2$
4. $P_5P_6 = \varphi P_1P_2$
5. $P_7P_8 = P_1P_2/\varphi$
6. $P_9P_{10} = \sqrt{\varphi}P_1P_2$
7. $P_{11}P_{12} = P_1P_2/\sqrt{3}$
8. $P_{13}P_{14} = 3P_{11}P_{12}$
9. $P_{15}P_{18} = \varphi^2P_{11}P_{12}$
10. $P_{15}P_{11} = P_{12}P_{16} = P_{11}P_{12}/\varphi$
11. $P_{19}P_{20} = P_{11}P_{12}/\varphi^2$

**Proof.** The new statements (2, 7–11, and part of 1) may be verified relatively easily for the quartic $x^4 - 6x^2$, that is, $f(x)$ from (2) with $w = -6$. (Since then $f''(\pm 1) = 0$, this quartic is simpler to use for $P_{11}$ through $P_{20}$ than $x^4 - 2x^2$.) Now, inflection points are mapped to inflection points by an affine map. (Indeed, $g''(\bar{x}) = ak^2f''(x)$ in our case.) Then, the arguments made earlier apply here as well.

The properties of the line through the inflection points $P_{11}$ and $P_{12}$ (statement 10, and in part 1) have been pointed out earlier [2], as has the symmetry property (statement 2) and the fact that $P_0$ is the point where the tangent is parallel to the double
tangent (a consequence of statement 1) \([2, 3]\). I have found no reference to the other 
relations, including, in particular, the five occurrences of the golden ratio.

Our affine transformation (3) shows that the graph of a general quartic function may 
be regarded as an originally symmetric graph that has been sheared in the \(y\)-direction 
and moved. Considering this, the properties regarding parallelism and symmetry (for 
instance, that the line segments \(P_{15}P_{11}\) and \(P_{12}P_{16}\) have equal length) become obvious. 
The same applies to ratios between areas, since a scaling transformation changes all 
areas by a constant factor, while a shear preserves areas. For example, it is known that 
the line through the inflection points of an arbitrary quartic function cuts off three areas 
that are in the ratio of \(1 : 2 : 1\). This is readily verified for \(f(x)\) from (2) with \(w = 
-6\) by checking that \(\int_0^{\sqrt{5}} (f(x) - (-5)) \, dx = 0\), whereby it is immediately proven 
generally.

Quartic polynomials and pentagrams  Returning to FIGURE 1, we see that it is just 
an example of statements 4 and 5 of the theorem. The same can be illustrated by FIG-
URE 5a, where the smaller pentagram fits exactly into the inner pentagon of the larger 
pentagram (meaning the linear size ratio is \(1 : \phi^2\)). Similarly, as the reader may check, 
FIGURE 5b illustrates statements 9, 10, and 11. In each of these graphs, three points 
are given, two of which are specified as minimum points, (a), or inflection points, (b). 
This completely determines the graphs; they will automatically pass through four or 
six more vertices. The possibility of finding such simple constellations of pentagrams 
and quartic graphs reflects the occurrence of the golden ratio in quartic polynomials.

\[ \int_0^{\sqrt{5}} (f(x) - (-5)) \, dx = 0, \]

![Figure 5](image_url)  Quartic polynomials passing through pentagram vertices

To summarize, we have found simple characteristic length ratios on the graph of a 
quartic polynomial with inflection points, including several occurrences of the golden 
ratio. These length ratios are left invariant by an affine transformation that relates a 
symmetric quartic to a general quartic with inflection points.

REFERENCES