

Transitions to Proof

Carol Schumacher, Kenyon College (Chair)
Susanna Epp, DePaul University
Danny Solow, Case Western Reserve University

General Information

The *Title of the Course* varies widely. We encountered: *Introduction to Proof Techniques*; *Introduction to Higher Mathematics*; *Introduction to Abstract Mathematics*; *Foundations of Mathematics*; *Foundations, Fundamentals of Mathematics*; *Communicating Mathematics*; *Introduction to Mathematical Reasoning*.

Some institutions teach the course in the context of specific subject matter and the course title reflects this emphasis. *Discrete Mathematics* is quite common. Other titles such as *Linear Algebra*; *Introduction to Number Theory*; *Sequences, Series and Foundations*; *Laboratories in Mathematical Experimentation*; are less widespread but present.

Credit Hours per semester: usually a 3-credit course.

Target student audience: The course is aimed at mathematical sciences majors or minors---usually sophomores or advanced first year students who have completed some introductory college-level math courses and are ready to progress to abstract, higher-level mathematics courses.

Prerequisites: Again, there are many possibilities. The prerequisites are usually “mathematical maturity” prerequisites, rather than content-based prerequisites. Calculus II seems to be the most common choice. But there is a range that stretches from only Calculus I through Linear Algebra or Multivariable Calculus. There is a trade-off between getting students doing serious mathematics very early in their college career and giving students a bit of time to mature intellectually before the transitions course. This is a decision best made with knowledge of the local student population and in the context of more general curricular considerations.

History and Unifying Themes

The cognitive course goals assert that “every mathematical sciences major [should] help students acquire ‘mathematical habits of mind.’ Students should develop the ability and inclination to use precise language, reason carefully, solve problems effectively, and use mathematics to advance arguments and increase understanding.” Moreover, programs should, among other things, “promote students’ progress in . . . understand[ing] the importance of precise definition; deduce general principles from particular instances; assess the correctness of solutions, create and explore examples, carry out mathematical experiments, and devise and test conjectures; recognize and make mathematically

rigorous arguments; communicate mathematical ideas clearly and coherently both verbally and in writing; approach mathematical problems with curiosity and creativity and persist in the face of difficulties; work creatively and self-sufficiently with mathematics.”

Such skills are crucial in mathematics and so these goals seem unobjectionable. Unfortunately, jam-packed syllabi can create a tension between the imperative to cover content and giving students *time* to wrap their minds around the mathematics in these important ways. All too often important cognitive goals give way to making sure our students “see” important mathematical ideas. As instructors, we may close our eyes and cross our fingers, hoping that our students are coming to grips with the details outside of class time. A few students do, picking up analytical and critical thinking skills by osmosis. Most students can’t, however, because they have no idea how to go about it, or (worse) don’t know what it means to do so. Such students can sometimes get through lower-level courses by imitation, but struggle in upper-level courses that require them to think abstractly, construct logical arguments, and use mathematical language precisely. It was this observation that led to the proliferation of so-called “transition” courses, which were rare in the early 1990’s but now are quite common. The primary purpose of a transition course is to ramp up students’ abilities to think and approach problems like mathematicians, providing a cognitive bridge between more procedural lower-level courses such as Calculus and upper-level abstract courses such as Real Analysis, Probability Theory, or Abstract Algebra. In transition courses, content goals take a back seat; the primary goals of the course are cognitive. Where time constraints cause tension between cognitive goals and content coverage goals, content should *always* give way to activities that help students progress in developing analytical, critical-reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind.

Transition courses are, of course, not devoid of mathematical content. If students are to reason carefully, think critically, solve problems, and communicate mathematical ideas precisely, they must have ideas to grapple with, problems to solve, and opportunities to talk and write about mathematics. However, the choice of mathematical “context” varies quite a bit. Many institutions teach a course centered on standard “mathematical building blocks” such as sets, relations, functions, and so forth; others introduce students to mathematical reasoning in the context of specific subject matter. Some elementary Discrete Mathematics courses, for instance, introduce students to conjecture and proof using simple counting techniques and elementary graph theory. Other institutions offer an elementary Number Theory course or structure their introduction to Linear Algebra as their students’ introduction to higher mathematics. Another interesting, but more unusual, approach is to introduce students to higher mathematics through mathematical experimentation that leads to conjecture and, finally, proof. Despite the differences, these courses have one important thing in common: the number of topics is deliberately kept to a minimum so that students can concentrate on developing careful use mathematical language, practice logical reasoning skills, and learn theorem-proving skills. The emphasis throughout the course is on process rather than on content. Moreover, work with written and oral communication of mathematical ideas is an essential part of the course.

Central Goals

The course should concentrate on training students in clear thinking and creative experimentation in the exploration of mathematical ideas. Because proof solidifies intuition into certainty, the course should also focus on the careful use mathematical language, logical reasoning and proof. The course should concentrate on imparting to students:

- the ability to read, understand, and construct proofs;
- the ability to write and speak about mathematics using precise mathematical language;
- an understanding the role of definitions in mathematics and being able to use (and possibly construct) them effectively;
- a basic understanding of elementary logical principles and proof techniques. (Examples include the proper use of logical connectives and quantifiers, negation of mathematical statements, the equivalence of a statement and its contrapositive, direct proof, proof by contraposition, proof by contradiction, and proof by induction.)
- an understanding of generalization and abstraction and their roles in mathematics;
- the ability to create visual images from written mathematics and vice versa;
- the ability to identify similarities and differences between mathematical objects. (E.g. what are the similarities and differences between the real numbers and the integers?)
- knowing how to capture the essential elements of intuitive mathematical objects in precise language that can make them subject to rigorous mathematical analysis (e.g. definitions and axiom systems) and understanding the importance of this process in mathematical discourse.

Moreover, students don't learn to do these things by watching someone else do them. In a large-scale study of university mathematics courses, Sandra Laursen et. al. [1] found "strong and consistent evidence about the dual importance of individual engagement and collaborative learning processes" for student learning outcomes. Quoting from the results of this study:

Student learning gains correlated statistically significantly with the fraction of class time spent doing student-centered activities (small group work, student presentation, computer work, and discussion), and anti-correlated with the fraction of time spent listening to instructors talk. Similar correlations were found for the relation of learning outcomes to the proportion of class time that was student- or instructor-led, and for variables that reflect how students and instructors interact and share responsibility for the course. Moreover, statistical modeling shows that the degree of student-centered class time was the strongest predictor of student learning as measured by our broadest learning indicator, survey learning gains.

This study gives important insight into good pedagogy for all mathematics courses. Unfortunately, student-centered activities take a great deal of time and may consequently be crowded out by the “coverage” imperatives of content-driven courses. (As students learn mathematics by doing mathematics, this is unfortunate, but it certainly happens.) However, as we have already noted, the most important goals of the transition course are cognitive goals. Therefore a well-constructed syllabus for a transition course should always be “lean” enough in terms of content that students are actively engaged in the material at every step of the way---both in class and outside of class.

Sample content lists

It is not really important what mathematical “context” is used to teach mathematical reasoning and proof. We emphasize again that the main imperative for the course is to give students many *opportunities* to come to grips with mathematical ideas and language and enough *time* to wrap their heads around the material in a way that leads to true ownership of the mathematical ideas. Nevertheless, it may be useful to have some examples of content covered in some transitions to proof courses. There are other possibilities.

Building blocks of Mathematics: Basic logical principles and proof techniques. Elementary set theory---including unions, intersections, and complements and the relations between them. Relations --- including orderings and equivalence relations. Functions --- including one-to-one and onto and inverse functions; function composition; images and inverse images of sets under a function. Infinite sets and Cardinality.

Introduction to Number Theory: Divisibility of integers, prime numbers and the fundamental theorem of arithmetic. Congruences --- including linear congruences, the Chinese remainder theorem, Euler’s ϕ -function, and polynomial congruences, primitive roots. *Other topics may be included as time permits.* Some possibilities are: Diophantine equations. Number theoretic functions. Approximation of real numbers by rational numbers.

Discrete Mathematics: Basic logical principles and techniques of direct and indirect proof. Properties of integers and rational numbers. Sequences, induction, and recursion. Elementary set theory. Properties of relations and functions. Introduction to graph theory and/or introduction to combinatorics and probability.

Mathematical Experimentation: A selection of self-contained modules from various areas of mathematics (e.g. number theory, graph theory, geometry, sequences and series, complex numbers, dynamical systems) that lead to experimentation, conjecture, proof, generalization.

Technology: the choice of whether to incorporate technology in the transition course seems very dependent on other choices made about how to structure the course. For instance, a department that introduces proof through discrete mathematics or number

theory may have students use computers to explore patterns and make conjectures (or not), whereas a course that is more focused on set theoretic building blocks may feel this is less useful. Likewise, a course that has students explore mathematics experimentally, may or may not incorporate computation as tool. Some institutions use the transitions course to introduce their students to the use of LaTeX. Instructors in more advanced courses can then expect students to be able to typeset sophisticated mathematics in class projects, senior theses, etc.

References:

1. Laursen, S., Hassi, M.-L., Kogan, M., Hunter, A.-B., & Weston, T. (2011). *Evaluation of the IBL Mathematics Project: Student and Instructor Outcomes of Inquiry-Based Learning in College Mathematics*. [Report prepared for the Educational Advancement Foundation and the IBL Mathematics Centers]. Boulder, CO: Ethnography & Evaluation Research, University of Colorado Boulder.

The full report can be found at

http://www.colorado.edu/eer/research/documents/IBLmathReportALL_050211.pdf

Resources

The following are lists of textbooks and other resources that might be useful for undergraduate courses that help students “transition to proof.”

Textbooks

Remark: The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of transitions courses. Please note that some of the books listed were written by the authors of this report.

Basic Building Blocks of Mathematics: The following texts support a transitions course built around a discussion of logic, sets, relations, and functions. Most also feature a treatment of cardinality.

1. Bloch, Ethan D., *Proofs and Fundamentals: a first course in abstract mathematics*, 2nd edition. Springer, 2011.

Comments: This book includes an extensive and detailed treatment of logical principles and proof techniques, including substantial narrative that discusses the ideas behind these principles and how they are used by mathematicians. Also includes chapters on selected fundamental topics in Algebra, Combinatorics, Analysis, and Number Theory.

2. Burger, Edward B., *Extending the Frontiers of Mathematics: Inquiries into Proof and Argumentation*, Wiley, 2008.

Comments: This book is structured as a long series of interconnected problems, made up of statements that may or may not be true---the instructions to the student are frequently to “prove and extend” or “disprove and salvage.” Thus it supports an inquiry-based approach, and particularly encourages students to probe and conjecture. The book includes chapters on selected topics in Number Theory, Discrete mathematics, Algebra, and Analysis.

3. Schumacher, Carol, *Chapter Zero: fundamental notions of abstract mathematics*, 2nd Edition, Addison-Wesley, 2001.

Comments: This book supports an inquiry-based approach. Thus it contains very few finished proofs, so it is structured as a long series of problems that are left for the students. On the other hand, it supports the students’ mathematical development by helping them explore the motivation that underlies the ideas and by giving them practical tips about proof techniques and the construction of arguments. Its discussion of logical principles and proof techniques is brief and informal. Also includes chapters on selected elementary topics in Number Theory, and the Real Number System.

4. Smith, Douglas., Maurice Eggen, and Richard St. Andre, *A Transition to Advanced Mathematics*, 8th edition, Cengage Learning, 2015.

Comments: This book lends itself to a course in which there is a mixture of lecture and inquiry. It has a fairly extensive introduction to logic and approaches this more formally than some other books. The number of exercises is large and varied. Includes chapters on selected elementary topics in Abstract Algebra and Analysis.

Specific content areas: The following texts are built around various possible content areas, as indicated by their titles. Both are written with the goal that they will also be useful for introducing students to proof.

1. Epp, Susanna, *Discrete Mathematics with Applications*, 3rd Edition, Cengage Learning, 2003.
2. Marshall, David C., Edward Odell, and Michael Starbird, *Number Theory through Inquiry*, MAA Textbooks, 2007.

Mathematical Experimentation---the following book supports a transitions course focused on experimentation that leads to conjecture and finally proof.

3. Mount Holyoke, *Laboratories in Mathematical Experimentation: A Bridge to Higher Mathematics*, Springer-Verlag, 1997.

Comment: This book is no longer in print, but the full text and information about the way the course is taught at Mount Holyoke can be found here.

https://www.mtholyoke.edu/acad/math/lab_experimentation

Other relevant books

These books discuss proof strategies, effective mathematical thinking, and problem-solving that can be helpful for students who are in a transition to proof course. Each might be useful as a supplementary text or for supplementary reading in any transitions course.

4. Burger, Edward B., and Michael Starbird. *The 5 Elements of Effective Thinking*, Princeton University Press, 2012
5. Exner, George R., *An Accompaniment to Higher Mathematics*, Springer Verlag, New York, 1996.
6. Polya, G., *How to Solve It: A New Aspect of Mathematical Method*, 2nd Edition. Princeton University Press, 1985.
7. Solow, Daniel, *How to Read and Do Proofs*, 5th Edition, Wiley, 2010.

Articles from Mathematics Education Research:

8. Alcock, L., & Weber, K. (2005), Referential and syntactic approaches to proof: Case studies from a transition course. In H. L. Chick & J. L. Vincent (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 33-40). Melbourne: PME.

Comments: Two students from a transition-to-proof course were interviewed as they attempted proofs. One student, Brad, took a referential approach, meaning he used examples to make sense of the concepts. The other student, Carla, took a syntactic approach, meaning she used definitions and logic, but did not use examples. Both made progress on the proofs, but exhibited different strengths and different difficulties. The authors conclude, "Both students seem to have an underdeveloped notion of how to use examples and syntax together to construct a proof."

9. Edwards, Barbara S. and Michael B. Ward, Surprises from Mathematics Education Research: Student (Mis)use of Mathematical Definitions, *American Mathematical Monthly*, 111(5), 411-424.

Comments: When Edwards reported that she had found that the definitions of “limit” and “continuity” were problematic for some of the real analysis students, Ward’s intuitive reaction was that those words were “loaded” with connotations from their nonmathematical use and from their less than completely rigorous use in elementary calculus. He said, “I’ll bet students have less difficulty or, at least, different difficulties with definitions in abstract algebra. The words there, like ‘group’ and ‘coset,’ are not so loaded.” So the authors decided to study student understanding and use of definitions in Ward’s own introductory abstract algebra. Ward was surprised to see his algebra students having difficulties very similar to those of Edwards’s analysis students. In particular, he was surprised to see difficulties arising from the students’ understanding of the very nature of mathematical definitions, not just from the content of the definitions. This article reports their findings.

10. Epp, S. S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (Chapter 16). Reston, VA: NCTM.

Comments: While ostensibly directed at teachers of grades K-12, this is the story of how Epp came to understand her own university students’ difficulties with quantifiers, and how she changed her teaching as a result. It is replete with down-to-earth teaching examples.

11. Epp, S. S. (2009). Proof issues with existential quantification. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, Vol. 1. (pp. 154-159). Taipei, Taiwan: The Department of Mathematics, National Taiwan Normal University.

Comments: In this short paper, Epp discusses issues connected with the use of existential quantification in mathematics proofs. Examples of common incorrect proofs from university students are given, and issues raised by the proofs are analyzed: (1) the use of bound variables as if they continue to exist beyond the statements in which they are quantified, (2) the implicit use of existential instantiation, (3) the “dependence rule” for existential instantiation, and (4) universal instantiation and its use with existential instantiation. Suggestions for responding to student errors are offered.

12. Iannone, P., Inglis, M., Mejia-Ramos, J. P., Simpson, A. & Weber K. (2010). Does generating examples aid proof production? *Educational Studies in Mathematics*, 77, 1-14.

Comments: Many mathematics education researchers have suggested that asking learners to generate examples of mathematical concepts is an effective way of learning about novel concepts. To date, however, this suggestion has limited empirical support. Undergraduate students were asked to study a novel concept by either tackling example generation tasks, or reading worked solutions to these tasks. However, there was no advantage for the example generation group on subsequent proof production tasks. The authors found that undergraduate students overwhelmingly adopt a trial-and-error approach to example generation, and suggest that different example generation strategies may result in different learning gains. The authors conclude by stating that the teaching strategy of example generation is not yet understood well enough to be a viable pedagogical recommendation.

13. Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2011). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, 79(1), 3-18.

Comments: The authors consider what it means to comprehend a proof and how to assess that comprehension, by means other than simply having students reproduce a proof from memory. They consider how a proof is understood in terms of the meaning, logical status, and logical chaining of its individual statements, but also in terms of the proof's high-level ideas, its main components, the methods it employs, and how it relates to specific examples. The authors illustrate how each of these types of understanding can be assessed using a proof in number theory.

14. Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27, 249-266.

Comments: This is the first math education research study of a transition-to-proof course. The author observed the course and interviewed the students. He analyzed the data using the ideas of concept image, concept definition, and concept usage. He found seven kinds of student difficulty: the students did not know the definitions; they had little intuitive understanding of the concepts; their concept images were inadequate for doing proofs; students were unwilling or unable to generate examples; they did not know how to use the statement of a theorem to structure its proof; they were unable to understand and use mathematical language and notation; and they did not know how to get started on writing a proof.

15. Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.

Comments: The authors report on an exploratory study of the way eight mid-level undergraduate mathematics majors read and reflected on four student-generated arguments purported to be proofs of a single theorem. The results suggest that mid-level undergraduates tend to focus on surface features of such arguments and that their ability to determine whether arguments are proofs is very limited -- perhaps more so than either they or their instructors recognize. They begin by discussing arguments (purported proofs) and *validations* of those arguments, that is, reflections of individuals checking whether such arguments really are proofs of theorems. The authors provide a detailed analysis of the four student-generated arguments and finally analyze the eight students' validations of them.

16. Selden, A. & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. P. Carlson and C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 95-110), MAA Notes Volume No. 73. Washington, DC: Mathematical Association of America.

Comments: This chapter provides an overview of students' difficulties in learning to understand and construct proof. The major sections are titled: the curriculum and students' and teachers' conceptions of proof, understanding and using definitions and theorems, knowing how to read and check proofs, knowing and using relevant concepts, bringing appropriate knowledge to mind, knowing what's important and useful, and teaching proof and proving.

17. Tall, D. (1998). The cognitive development of proof: Is mathematical proof for all or for some? In Z. Usiskin (Ed.), *Developments in school mathematics education around the world* (Vol. 4, pp. 117-136). Reston, Virginia: NCTM.

Also:

<http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html>

Comments: Proof is often difficult to teach. In this paper, Tall suggests that different forms of proof are appropriate in different contexts, dependent on the particular forms of representation available to the individual, and that these forms become available at different stages of cognitive development. For a young child, proof may be a physical demonstration, long before sophisticated use of the verbal proofs of Euclidean geometry can be introduced successfully to a subset of the school population. Later still, formal proof from axioms involves even greater difficulties that make it

appropriate for a few, but impenetrable to many. At this formal stage of development, Tall identifies two different strategies that students adopt to come to terms with formal definition and deduction. Either strategy may be successful, but both are cognitively demanding and prove difficult for many to achieve. This leads to the observation that formal proof is appropriate only for some, that some forms of proof may be appropriate for more, and that, if one allows the simpler representations of proof such as those using physical demonstrations, perhaps some forms of proof are appropriate for (almost) all.

18. Weber, Keith, Students' Difficulties with Proof. MAA Online: Research Sampler. No 8, June 2003.
<http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof>

Comments: Weber discusses what is meant by the word “proof,” in various contexts, and the role that proof plays in mathematics. With this backdrop, he discusses the difficulties that many students experience with learning to prove theorems. Finally he makes some suggestions about how to effectively teach students the concept of proof. The paper is rich in additional references from the literature.