Long-term Behavior of Solutions to a Wave Equation with Degenerate Damping

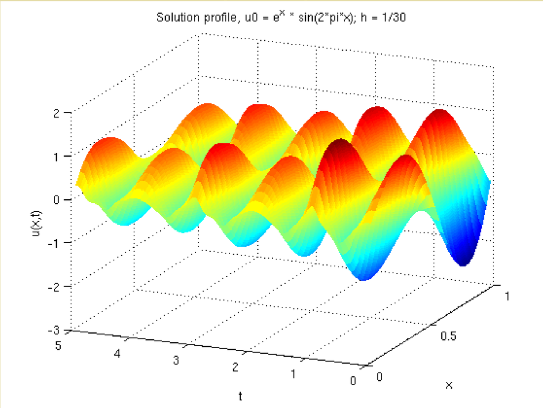
Thomas Anderson, Elizabeth Galvin, Ian Kessler, Michelle Kleckner, William Tritch
Advisors: Dr. George Avalos, Dr. Daniel Toundykov, Thomas Clark

Introduction

Goal: to investigate numerically the stability (strong and, if possible, uniform) of the 1D wave equation with "degenerate" nonlinear damping and derive theoretical results that support numerical approximations.

We studied the initial boundary value problem:

$u_0 = e^x \cdot \sin(2\pi x)$, $u_1 = 0$



$$\begin{cases} u_{tt} - u_{xx} + u^2 \cdot u_t = 0, \quad 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0, \quad \forall t \ge 0 \\ u(0,x) = u_0(x) \in V_1 = H_0^1(0,1) \\ u_t(0,x) = u_1(x) \in L^2(0,1) \end{cases}$$

State space: $H_0^1(0,1) \times L^2(0,1)$. **Norms:** $\|u\|_{L^2}^2 = \int_0^1 u^2(x)dx$ and for functions that are zero on the boundary $\|u\|_{H^1} \sim \|u_x\|_{L^2}$.

Local Existence

Theorem 1: Local Existence and Uniqueness of Weak Solutions

For $u_{tt} - u_{xx} + u^2 \cdot u_t = 0$ and its respective initial and boundary conditions, there is a time T such that a unique weak solution exists for $t \in [0, T]$.

Proof Outline: The solution to $u_{tt} - u_{xx} + u^2 \cdot u_t = 0$ satisfies:

$$\vec{y}(t) = S(t)\vec{y}_0 + \int_0^t S(t-s)\vec{F}(\vec{y}(s))ds$$

Lemma: Given (1) $R > \|\vec{y}_0\|_{H^1xL^2}$

(2) $\Lambda_T \vec{y}^{(n)} = S(t)\vec{y}_0 + \int_0^t S(t-s)\vec{F}(\vec{y}^{(n)})ds$

(3) $B_R = \{f : \|f\|_x \le R\}$ where $\|f\|_x = \max_{t\in[0,T]} \|f\|_{H^1xL^2}$

...then Λ_T is a contraction on the metric space B_R **if** R and T satisfy:

$$T < \min\{(R - \|\vec{y}_0\|_{H^1xL^2})/R^3,\ 1/3R^2\}$$

Therefore, $\{\Lambda_T^{(n)}(\text{initial guess})\}$ converges to the solution $\vec{y}$ on [0,T] by the **contraction mapping principle**.

Theorem 2: Local Existence / Uniqueness of Strong & Regular Solutions

Analagously, the Theorem 1 extends to show local existence of solutions in $(H^2 \cap H_0^1) \times H_0^1$ (strong solutions) and $H^3 \times H^2$ ("regular" solutions).

Global Existence

Theorem 3: Global weak solutions

There exists a weak solution to $u_{tt} - u_{xx} + u^2 \cdot u_t = 0$ for all time.

Proof Outline: **Finite Energy** $E(t) := \frac{1}{2}\|u_t(t)\|_{L^2}^2 + \frac{1}{2}\|u_x(t)\|_{L^2}^2$ satisfies $E(T) + \int_0^T \int_0^1 u^2 u_t^2 dx\, dt = E(0)$. Since $E(T)$ is non-increasing can invoke Theorem 1 to extend past any asserted maximal existence time.

Theorem 4: Global strong solutions

There exists a strong solution to $u_{tt} - u_{xx} + u^2 \cdot u_t = 0$ for all time.

Proof Outline: **Strong Energy** $\mathcal{E}(t) = \frac{1}{2}\|u_{tt}\|^2 + \frac{1}{2}\|u_{tx}\|^2$ can be shown to obey $\mathcal{E}(t) \le \mathcal{E}(0) + C(E(0))\int_0^t \mathcal{E}(s)ds$. Use Grönwall's inequality.

Use Finite Element Method to Solve a Linear Wave Equation

The weak formulation (in space) of the linear PDE $u_{tt} - u_{xx} = f(x,t)$ has the form

$$(u_{tt}, \psi) + (u_x(x,t), \psi'(x)) = (f(t), \psi) \text{ for all } \psi \in H_0^1(0,1)$$

Apply Rayleigh-Ritz method and approximate $u(x,t) \approx \sum_{i=1}^n \alpha_i(t)\phi_i(x)$ via piecewise linear nodal basis $\{\phi_i\}$ ("hat" functions centered at x_i). Then for $j = 1, \ldots, n$

$$\sum_{i=1}^n \alpha_i''(t)\Big(\phi_i(x), \phi_j(x)\Big) + \sum_{i=1}^n \alpha_i(t)\Big(\phi_i'(x), \phi_j'(x)\Big) = \Big(f(x,t), \phi_j(x)\Big)$$

Written as a matrix equation with $(M)_{i,j} = (\phi_i, \phi_j)$,$(A)_{i,j} = (\phi_i', \phi_j')$,
and then substituting $\vec{\beta} = \vec{\alpha}'$ we write the result as a system of ODEs:

$$M\vec{\alpha}''(t) + A\vec{\alpha}(t) = \vec{F}(t)$$

$$\frac{d}{dt}\begin{pmatrix}\vec{\alpha} \\ \vec{\beta}\end{pmatrix} + \begin{pmatrix} 0 & -1 \\ M^{-1}A & 0\end{pmatrix}\begin{pmatrix}\vec{\alpha} \\ \vec{\beta}\end{pmatrix} = \begin{pmatrix} 0 \\ M^{-1}\vec{F}(t)\end{pmatrix}$$

Get initial data by projecting u_0 and u_1 onto subspaces of $H_0^1(0,1)$ and $L^2(0,1)$:

$$\vec{\beta}(0) = M^{-1}\begin{pmatrix}(u_1, \phi_1) \\ \vdots \\ (u_1, \phi_n)\end{pmatrix}, \vec{\alpha}(0) = A^{-1}\begin{pmatrix}(u_0, \phi_1)_{H^1} \\ \vdots \\ (u_0, \phi_n)_{H^1}\end{pmatrix}$$

Solving the Nonlinear PDE using Successive Approximations

Since Λ_T is a contraction, we can use the method of successive approximations for the nonlinearity. Treat $-u^2 u_t$ as a known forcing term $f(x,t)$ and solve the now-linear problem.

Algorithm: Given previous solution $u^{(i)}(x,t)$, form forcing term $f_i(x,t)$

$$u^{(i)}(x,t) \longrightarrow f^{(i)}(x,t) = -(u^{(i)}(x,t))^2 u_t^{(i)}(x,t),$$

and solve the wave equation with this forcing term to obtain $u^{(i+1)}(x,t)$:

$$\frac{\partial^2 u^{(i+1)}}{\partial t^2} - \frac{\partial^2 u^{(i+1)}}{\partial x^2} = -(u^{(i)}(x,t))^2 u_t^{(i)}(x,t).$$

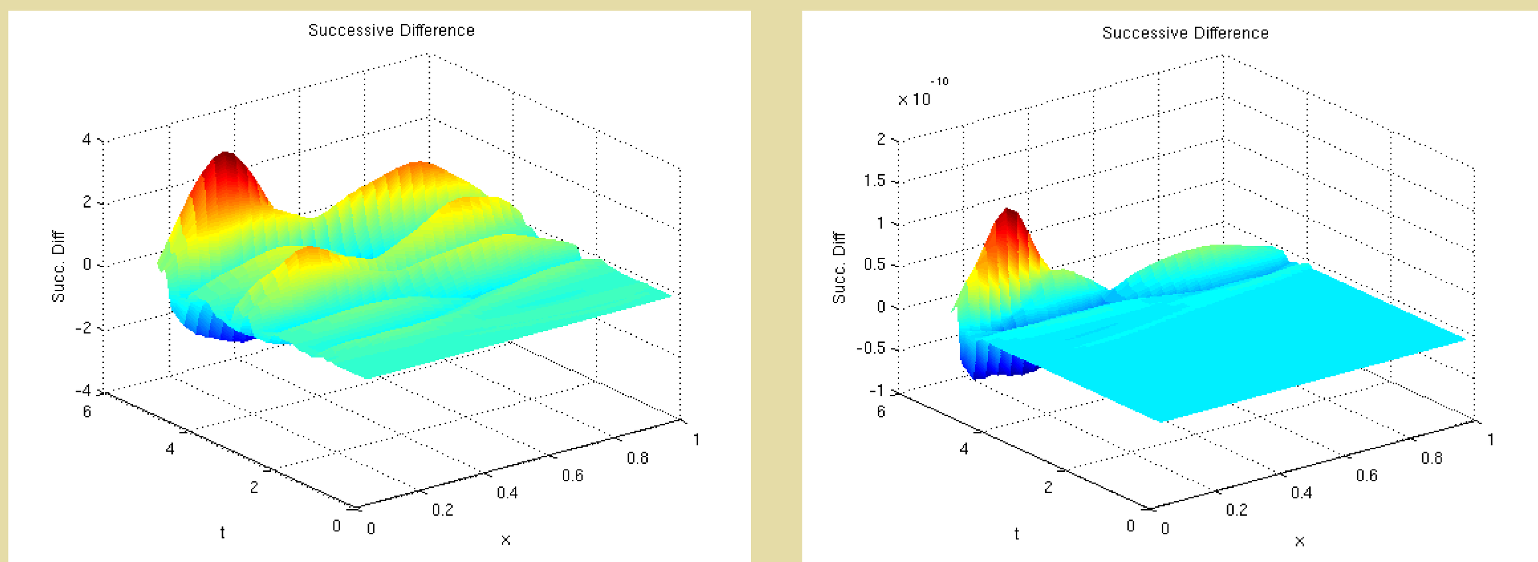
$$\frac{d}{dt}\begin{pmatrix}\vec{\alpha}^{(i+1)} \\ \vec{\beta}^{(i+1)}\end{pmatrix} + \begin{pmatrix} 0 & -1 \\ M^{-1}A & 0\end{pmatrix}\begin{pmatrix}\vec{\alpha}^{(i+1)} \\ \vec{\beta}^{(i+1)}\end{pmatrix} = \begin{pmatrix} 0 \\ M^{-1}\vec{F}^{(i)}(t)\end{pmatrix}$$

Knowing $u^{(i)}(x,t)$ and thus $\vec{F}^{(i)}(t)$ at only discrete times t, we are limited to ODE solvers which require only information at previous time steps. Linear multistep methods like the Adams-Bashforth class satisfy this:

$$\frac{d}{dt}\vec{y} = \vec{H}(t,y); \vec{y}_{n+1} = \sum_{j=0}^p a_j \vec{y}_{n-j} + \Delta t \sum_{j=0}^p b_j \vec{H}(t_{n-j}, \vec{y}_{n-j})$$

End the fixed point iteration when $|u^{(i+1)}(x,t) - u^{(i)}(x,t)| < tol, \forall t, x$.

Convergence of the Fixed Point Iterations



(a) Difference between iterations #1 & #2 (b) Difference between iterations #18 & #19

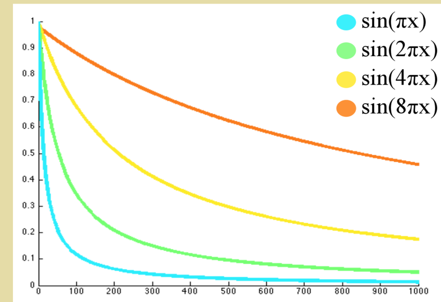
Figure : Plots of difference in solutions of successive iterations

Stability for the ODE prototype $w''(t) + kw(t) + w(t)^2 w'(t) = 0$

- Asymptotic (strong) stability: $\forall y_0 \in \mathbb{R}^2$, $\lim_{t\to\infty} y(t) = \mathbf{0}$ follows by LaSalle's invariance principle via Lyapunov function $V(y) = \frac{1}{2}y_1^2 + \frac{k}{2}y_2^2$
- Uniform stability: for bounded $B \subset \mathbb{R}^2$ and $\varepsilon > 0$ there is $T_{\varepsilon,B}$ s.t. $y_0 \in B$, $t \ge T_{\varepsilon,B}$, implies $y(t) \in B_\varepsilon(\mathbf{0})$. Follows from strong stability, continuous dependence on initial data, and Lyapunov function V.

Decay of the Energy of PDE Solutions as $t \to \infty$

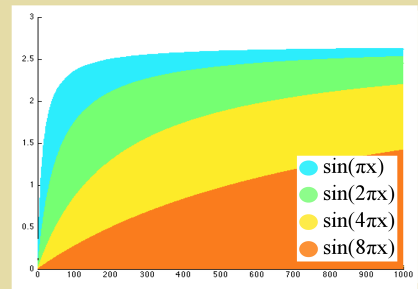
Energy vs. time. Conjecture is that solutions decay uniformly at a logarithmic rate. We can prove a partial uniform stability result in a lower norm.



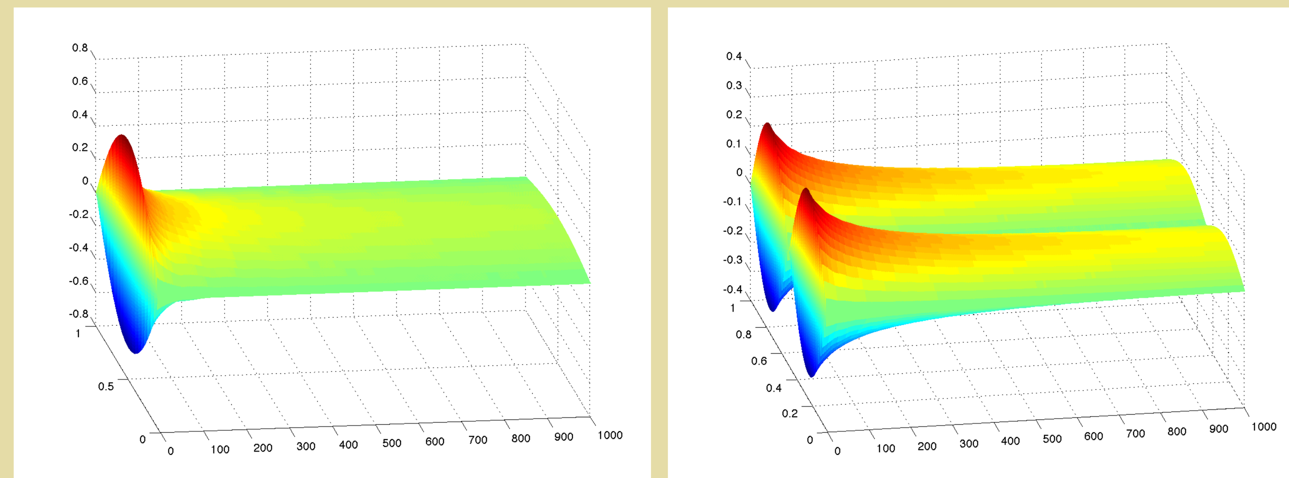
Theorem 5: [LT93]

If $z_{tt} + z_{xx} + \frac{1}{3}z_t^3 = 0$, then $\|z_t(t)\|_{L^2}^2 + \|z(t)\|_{H^1}^2 \sim Ct^{-1}$ as $t \to \infty$.

Then $u = z_t$ solves $u_{tt} - u_{xx} + u^2 u_t = 0$ and $\|u\|_{L^2}^2 = \|z_t\|_{L^2}^2$ decays as Ct^{-1}. Conjecture that this may hold for all solutions.



Numerical Solution Decay for $u_{tt} - u_{xx} + u^2 u_t = 0$



(a) $u_0 = \sin \pi x \quad u_1 = 0$ (b) $u_0 = \sin 2\pi x \quad u_1 = 0$

Future Work

Strong stability of $u_{tt} - u_{xx} + u^2 u_t = 0$ would follow for trajectories that are compact in $H_0^1 \times L^2$. E.g. if bounded in $H^2 \times H^1$. So far Theorem 4 yields $\mathcal{E}(t) \le \mathcal{E}(0)\exp(C \cdot E(0)t)$ for strong solutions. Proving a global bound is still an open question.

References

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