Long-term Behavior of Solutions to a Wave Equation with Degenerate Damping

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Introduction

Goal: to investigate numerically the stability (strong and, if possible, uniform) of the 1D wave equation with “degenerate” nonlinear damping and derive theoretical results that support numerical approximations.

We studied the initial boundary value problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} & = 0, \quad 0 \leq x < 1 \\
\frac{\partial u}{\partial t} \bigg|_{x=0} & = 0, \\
\frac{\partial u}{\partial t} \bigg|_{x=1} & = 0, \\
\frac{\partial u}{\partial t} \bigg|_{t=0} & = u_0(x), \\
u(0,x) & = u(0,x),
\end{align*}
\]

We assume for simplicity that \( u_0 \) is a strong solution to

\[
\frac{\partial^2 u_0}{\partial x^2} = 0.
\]

### Proof Outline

There exists a strong solution to the initial boundary value problem.

Theorem 1: Local Existence and Uniqueness of Weak Solutions

\[\exists u_0 \in H^1(0,1) \times L^2(0,1) \text{ such that a unique weak solution exists for } u(t,x).\]

Theorem 2: Local Existence / Uniqueness of Strong & Regular Solutions

\[\exists u_0 \in H^1(0,1) \times L^2(0,1) \text{ such that a unique strong solution exists for } u(t,x).\]

Local Existence

Theorem 1: Local Existence and Uniqueness of Weak Solutions

For \( u_0 \in H^1(0,1) \times L^2(0,1) \) and its respective initial and boundary conditions, there is a time \( T \) such that a unique weak solution exists for \( t \in [0,T] \).

Proof Outline: The solution to \( u_{tt} - \Delta u = 0 \) satisfies:

\[\begin{align*}
\partial y(t)^2 & = 0, \\
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\end{align*}\]

Lemma: Given \( 1 > \partial y(t)^2 \)

\[\begin{align*}
\partial y(t)^2 & = 0, \\
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\end{align*}\]

\[\partial y(t)^2 \] is a contraction on the metric space \( R^{3/2} \).

Therefore, \( \{y(t)^2\} \) converges to the solution \( y \) on \([0,T]\) by the contraction mapping principle.

### Global Existence

Theorem 3: Global weak solutions

There exists a weak solution to \( u_{tt} - \Delta u = 0 \) for all time.

Proof Outline: Finite Energy

\[ E(t) := \frac{1}{2} \int_0^1 |u(t,x)|^2 + \frac{1}{2} \int_0^1 |\partial_t u(t,x)|^2 |dx| = E(0). \]

Since \( E(T) \) is non-increasing, we invoke Theorem 1 to extend past any asserted maximal existence time.

Theorem 4: Global strong solutions

There exists a strong solution to \( u_{tt} - \Delta u = 0 \) for all time.

Proof Outline: Strong Energy

\[ E(t) = \frac{1}{2} \int_0^1 |u(t,x)|^2 + \frac{1}{2} \int_0^1 |\partial_t u(t,x)|^2 |dx| \]

is shown to obey \( E(t) \leq E(0) + C(E(0)) \int_0^t E(s) |ds| \) by use of Gronwall’s inequality.

Use Finite Element Method to Solve a Linear Wave Equation

The weak formulation (in space) of the linear PDE \( u_{tt} - \Delta u = f(x,t) \) has the form

\[ (u_0, v) + (u_1, v) + \int_0^1 f(t,x) v |dx| = 0 \]

Apply Rayleigh-Ritz method and approximate \( u(t,x) \approx \sum_1^n \phi_1(x) \phi_1(x) \) via piecewise linear nodal basis \( \{\phi_i\} \) (“hat” functions centered at \( x_i \)). Then for \( j = 1, \ldots, n \)

\[ \sum_1^n \alpha_i(t) \phi_i(x) \cdot \phi_i(x) + \sum_1^n \alpha_i(t) \phi_i(x) \cdot \phi_i(x) = \int_0^1 f(t,x) \phi_i(x) \phi_i(x) \]

Written as a matrix equation with \( (M)_{ij} = (\phi_i,\phi_j), (A)_{ij} = (\phi_i,\phi_j) \), and then substituting \( \beta = \beta^\prime \) we write the result as a system of ODEs:

\[ M\beta(t) + A\beta(t) = f(t) \]

\[ \frac{d}{dt} \beta(t) + (M^{-1} A \beta(t)) = f(t) \]

Get initial data by projecting \( u_0 \) onto subspaces of \( H^1(0,1) \) and \( L^2(0,1) \):

\[ \beta(0) = M^{-1} \left( (u_0,\phi_1), \ldots, (u_0,\phi_n) \right), \beta(0) = A^{-1} \left( (u_0,\phi_1), \ldots, (u_0,\phi_n) \right) \]

Solving the Nonlinear PDE using Successive Approximations

Since \( A_T \) is a contraction, we can use the method of successive approximations for the nonlinearity. Treat \( u_{tt} \) as a known forcing term \( f(x,t) \) and solve the non-linear problem.

Algorithm:

1. Given \( \beta^{(0)} \) solve \( \beta^{(1)} \) to obtain \( u^{(1)} \) such that

\[ u^{(1)} = \sum \beta^{(1)}(x) \phi_1(x) \]

2. Then solve \( u^{(2)} \) to obtain \( u^{(2)} = \sum \beta^{(2)}(x) \phi_1(x) \)

3. ... and so on until the \( N \)th iterate of \( u^{N}(x) \) converges to the solution \( u(x) \).

Convergence of the Fixed Point Iterations

Prove the convergence of the fixed point iterations

\[ u^{N}(x) \rightarrow u(x) \]

Numerical Solution Decay for \( u_{tt} - \Delta u + \beta u_t = 0 \)

\[ (a) u_0 = \sin \pi x, \quad u_t = 0 \]

\[ (b) u_0 = \sin 2\pi x, \quad u_t = 0 \]

Future Work

Strong stability of \( u_{tt} - \Delta u + \beta u_t = 0 \) would follow for trajectories that are compact in \( H^1 \times L^2 \). E.g. if bounded in \( H^1 \times L^2 \). So far Theorem 4 yields

\[ E(t) \leq C \int_0^t E(s) \]

for strong solutions. Proving a global bound is still an open question.

### References