References

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Teaching Tip: Correcting Cramer's Rule

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There is a misunderstanding about Cramer's rule that examination of more than a dozen intermediate algebra texts (including [**1**], [**2**], and [**3**]), suggests few have avoided. The error also appeared in Wikipedia [**4**]. It has been corrected by the author.

Let D be the determinant of a linear system of equations and D_x, D_y, D_z, etc. be determinants formed by replacing in turn each column of D by the constants of the system. The sources cited all contain the following incorrect statement (with some unimportant differences in wording): for a 3×3 system of linear equations, if all determinants are zero, then the system has infinitely many solutions. What this statement doesn't take into account is that besides infinitely many solutions, the system may be inconsistent and have *no* solution, for example

$$\begin{cases} x + 2y + 3z = 4 \\ x + 2y + 3z = 5 \\ x + 2y + 3z = 6. \end{cases}$$

Geometrically this system represents three parallel planes.

Table 1.

<table>
<tr><th>System</th><th>Solution</th><th>Kronecker-Capelli Theorem</th><th>Cramer's Rule</th></tr>
<tr><td rowspan="3">2×2</td><td>US</td><td>$r = r' = 2$</td><td>$D \neq 0 \quad x = D_x/D, y = D_y/D$</td></tr>
<tr><td>NS</td><td>$r = 1, \quad r' = 2$</td><td>$D = 0, D_x \neq 0, D_y \neq 0$</td></tr>
<tr><td>IMS</td><td>$r = 1, \quad r' = 1$</td><td>$D = 0, D_x = 0, D_y = 0$</td></tr>
<tr><td rowspan="5">3×3</td><td>US</td><td>$r = r' = 3$</td><td>$D \neq 0 \quad x = D_x/D, y = D_y/D, z = D_z/D$</td></tr>
<tr><td rowspan="2">NS</td><td>$r = 1, 2, \quad r' = 3$</td><td>$D = 0, D_x^2 + D_y^2 + D_z^2 > 0$</td></tr>
<tr><td>$r = 1, \quad r' = 2 \quad *$</td><td rowspan="3">$D = 0, D_x = 0, D_y = 0, D_z = 0$</td></tr>
<tr><td rowspan="2">IMS</td><td>$r = r' = 2$</td></tr>
<tr><td>$r = r' = 1$</td></tr>
</table>

The situation is clarified by the Kronecker-Capelli theorem. Let A be the matrix of coefficients of a system of linear equations; let A' be its augmented matrix; and let r and r' be the ranks of these matrices. The Kronecker-Capelli theorem states that the system has a solution only if $r = r'$. The system has no solution if $r < r'$. The extension of the theorem is that the solution is unique if $r = r' = n$, and there are infinitely many solutions if $r = r' < n$.

Ignoring the case $r = 0$, Table 1 lists all cases (US = unique solution, NS = no solution, and IMS = infinitely many solutions) and their connection with Cramer's Rule. The error-prone, ambiguous case is marked by an asterisk.

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Flaws, Fallacies, and Flimflam: Who's Right?

The following items appeared in the *Atlantic Monthly* for March, 2010 in the Letters Column (page 17). Is either one of these contributions correct?

Misleading Math?

> Megan McArdle attempts to illustrate her point that survey respondents are unreliable ("Misleading Indicator," *November Atlantic*) by telling us that it is mathematically impossible for men to report an average number of female sexual partners that is much higher than the average number of male partners reported by women. I agree that survey respondents are unreliable, but so is McArdle's math. It may be unlikely that the number of partners reported by honest males would be higher, but it is not mathematically impossible.
>
> —Fred Graf, Concord, NH

Megan McArdle replies:

> We are talking about two different meanings of the word average. True, in Harvey Levin's scenario, it is more common for women to be monogamous than men. But I was using average to describe the mean number of sexual partners, which is to say, the sum of everyone's number of partners, divided by the number of people. If there are 10 men and 10 women, and one of the women has slept with all 10 men, while the other women are monogamous, the average number of sexual partners is the same for the men and the women. Nine of the men have had two sexual partners, while one of the men has had one partner, for an average of $19/10 = 1.9$. Meanwhile, one of the women has had 10 sexual partners while the other nine women have had one partner, for an average of $19/10 = 1.9$. The numbers come out the same, no matter how you vary the particulars. Yet surveys can generate differences of three- or fourfold between the mean numbers of sexual partners that women and men report.

—Ed Barbeau (barbeau@math.utoronto.ca), University of Toronto