THE SUBTREE POLYNOMIAL: A GENERATING FUNCTION ON GRAPHS

BY: Alex Chin[1], Kellie MacPhee[2], and Charles Vincent[3]

ADVISOR: Gary Gordon[3]

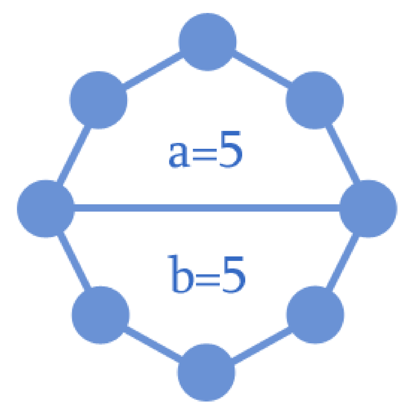
[1] North Carolina State University, Raleigh, NC

[2] Dartmouth College, Hanover, NH

[3] Lafayette College, Easton, PA

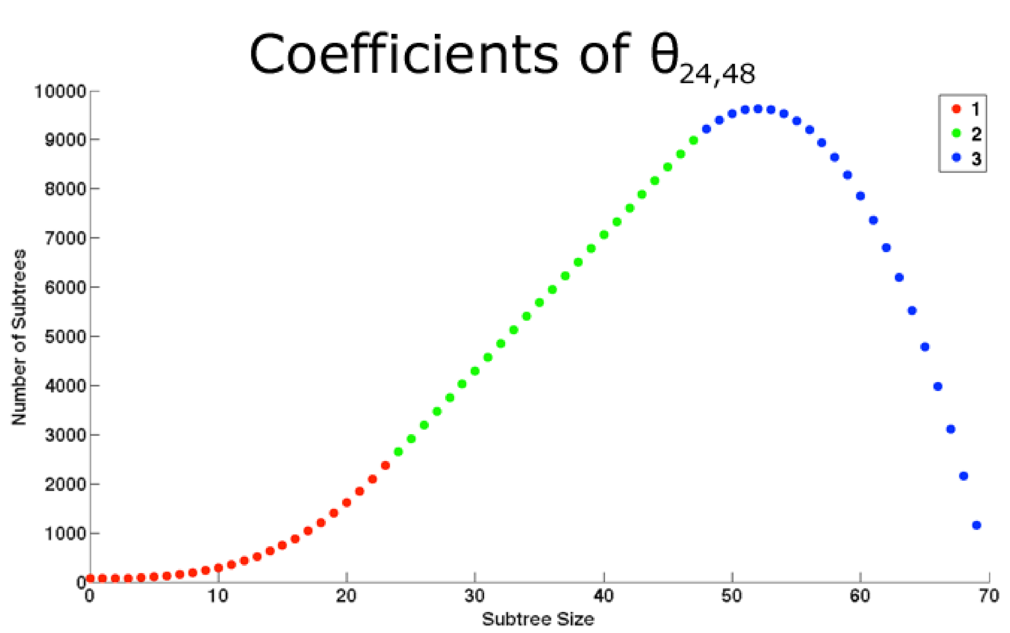
THETA GRAPHS

Theta graphs are the class of graphs which contain one more edge than a cycle. We denote them by $\theta_{a,b}$, where a is the size of the upper cycle and b is the size of the lower cycle. As an example, the theta graph shown to the right would be a $\theta_{5,5}$.



FACT: The subtree polynomial for $\theta_{a,b}$ is given by the formula: $F(\theta_{a,b})=n\sum_{k=0}^{n-1}x^k+x\left(\sum_{i=0}^{a-2}(i+1)x^i\right)\left(\sum_{j=0}^{b-2}(j+1)x^j\right)$

GRAPHING THE COEFFICIENTS



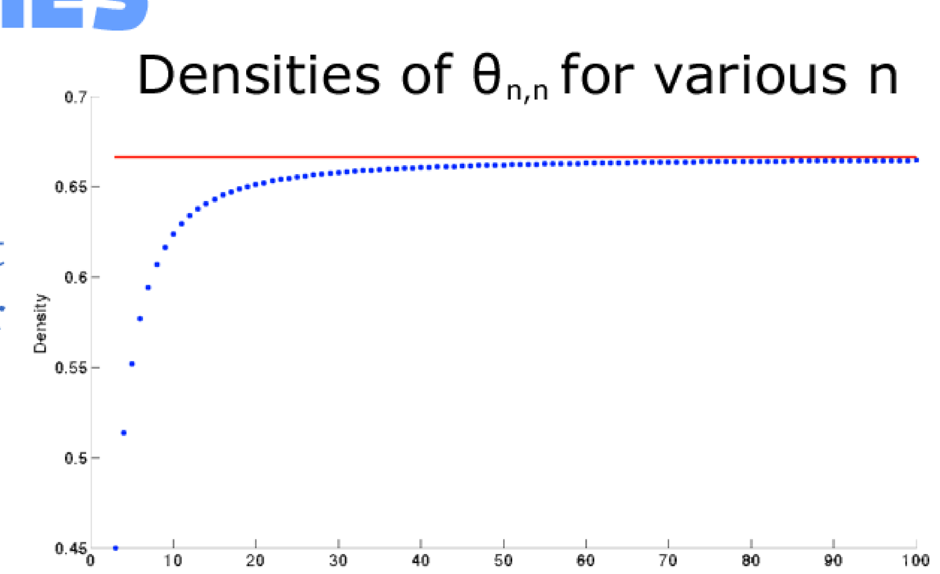
A graph of the coefficients of the subtree polynomial for $\theta_{24,48}$ is shown to the left. The x-axis is the size of the subtree (i.e., how many edges it contains) and the y-axis is the number of subtrees of that size.

This graph appears to be piecewise, made up of three distinct parts. Indeed, analysis on our generalized formula confirms that all theta coefficient graphs exhibit this behavior. Specifically, these coefficients follow a cubic function in the beginning, then become linear before following a cubic again. From this, we have the following:

FACT: The maximum coefficient in the polynomial for $\theta_{a,b}$ occurs at: $-1+\sqrt{(a-1/2)^2+(b-1/2)^2-1/6}$

THETA DENSITIES

Through graphical experimentation, we found that the density for larger and larger theta graphs seems to approach a curious limit: 2/3. We have managed to prove this as well.



THEOREM: $D(\theta_{n,n})\longrightarrow 2/3$

THE POLYNOMIAL ITSELF

DEFINITION:

The subtree polynomial of a graph G is:

$$F(G)=k_0+k_1x+k_2x^2+\dots+k_{n-1}x^{n-1}=\sum_{i=0}^{n-1}k_ix^i$$

where k_i is the number of subtrees of G with i edges.

BASICS OF GRAPH THEORY

A graph is a set of vertices and a set of edges such that the edges can be described as a pair of vertices.

A tree is a graph which is both connected and cycle-free.

A graph is a cycle when it contains one additional edge and every vertex is contained in only two edges.

SUBTREE DENSITY

DEFINITION:

The average subtree size of a graph G is:

$$\mu(G)=\frac{\#\text{ edges used in subtrees}}{\#\text{ subtrees}}=\frac{F'(1)}{F(1)}$$

DEFINITION:

The subtree density of a graph G is: $D(G)=\dfrac{\mu(G)}{n-1}$

This is a real number between zero and one.

OPEN QUESTIONS

What is the mode subtree size of other classes of graphs?

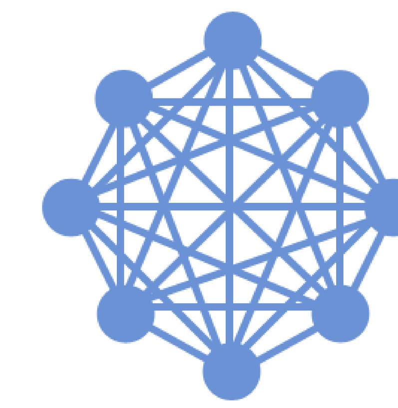
The subtree density?

The proportion of spanning subtrees?

Can we prove that $\lim_{n\to\infty}D(K_{n,n})\longrightarrow 1$?

How many edges, as a fraction of ${}_nC_2$, are required for a sequence of graphs G_n to be such that $\lim_{n\to\infty}D(G_n)=1$?

COMPLETE GRAPHS



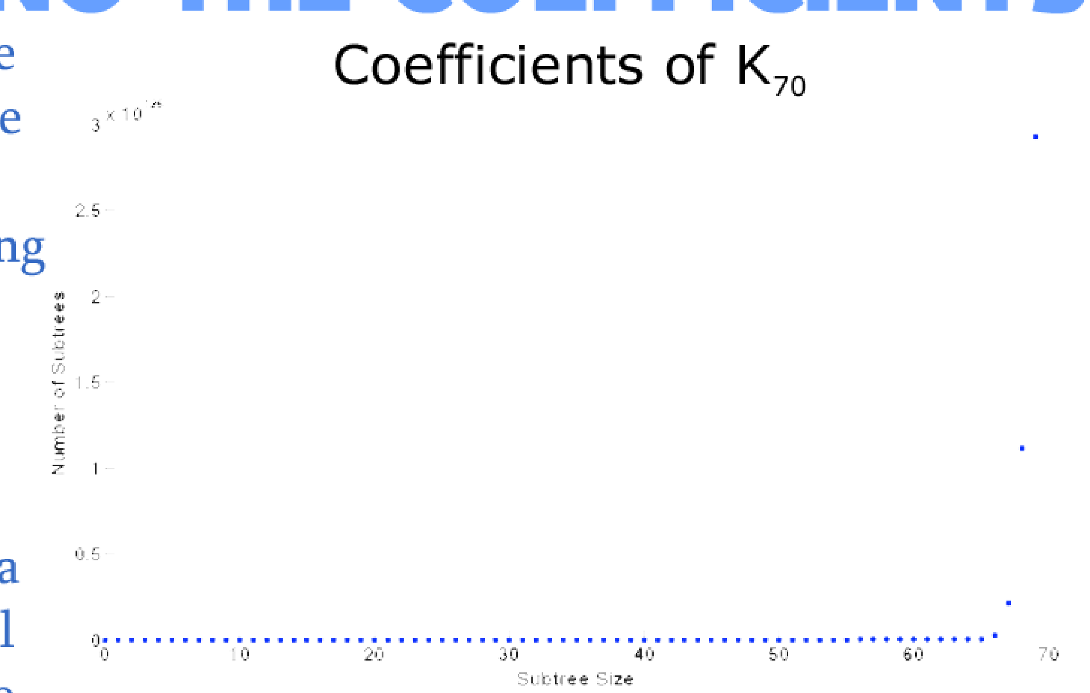
A complete graph, K_n, is a set of n vertices connected such that every possible pair of two distinct points is an edge. Pictured to the left is K_8, the complete graph on eight vertices. Since every vertex and edge is essentially the same, the subtree polynomial is relatively simple to generalize.

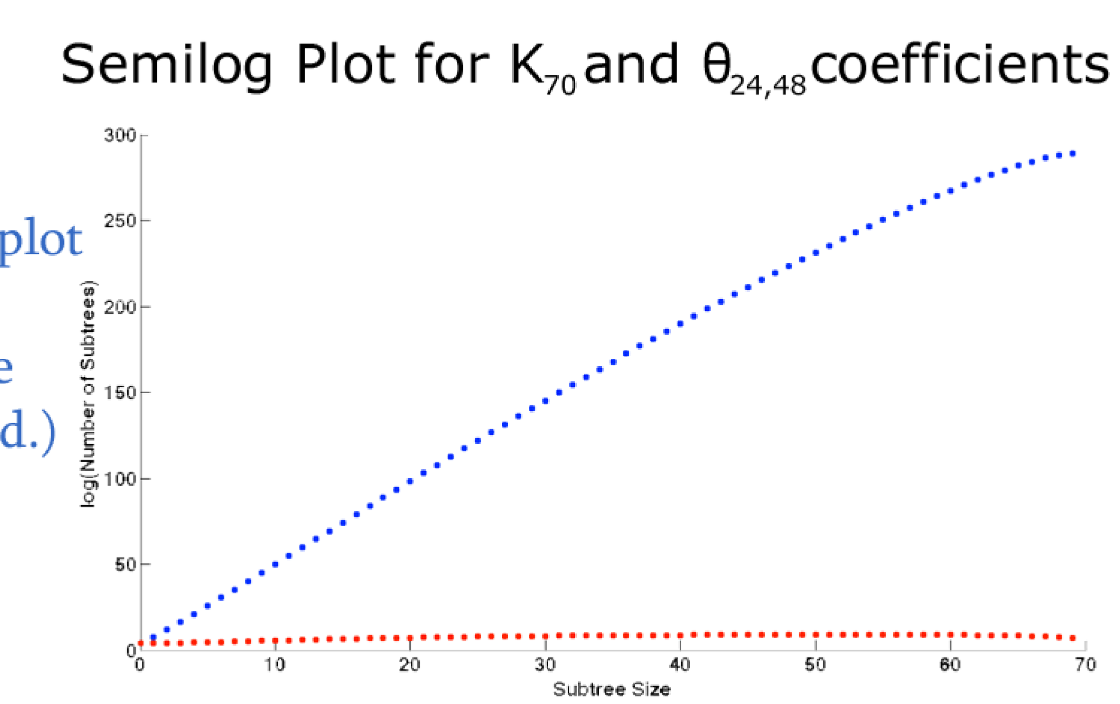
FACT: The subtree polynomial for K_n is: $F(K_n)=\sum_{i=0}^{n-1}{}_nC_{i+1}(i+1)^{i-1}x^i$

GRAPHING THE COEFFICIENTS

To the right is a plot of the coefficients of K_{70}, with the x-axis being the size of a subtree and the y-axis being the number of subtrees of the given size.

Almost all subtrees are the size of spanning trees. This is different from theta graphs, where only a small fraction of the subtrees are spanning.





To the right is a semilog plot of the coefficients of K_{70} (in blue) compared to the coefficients of $\theta_{24,48}$ (in red.) The polynomial for K_{70} quite clearly dominates over that of $\theta_{24,48}$.

COMPLETE DENSITIES

THEOREM: $D(K_n)\longrightarrow 1$

COROLLARY: The number of spanning trees of K_n that are spanning when n is large is $e^{-1/e}$ or approximately 69.2%