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Application of the Lambert *W* Function to the SIR Epidemic Model

Frank Wang (fwang@lagcc.cuny.edu), LaGuardia Community College, CUNY, Long Island City, NY 11101

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The SIR model for the spread of an epidemic was formulated by Kermack and McKendrick in 1927 [**1**]. It is still the basic framework around which most modern infectious disease models are constructed. The letters in the acronym represent the three classes that any member of a population can occupy with respect to a disease: *Susceptible*, *Infected*, and *Removed*. Although the SIR model is a simple nonlinear dynamical system, it admits no closed-form solution as explicit functions of time. In this Capsule we show that the stable fixed point—asymptotic values of the population in different classes—can be expressed in terms of the Lambert W function. We remark that a similar approach has been used independently by some authors in the mathematical biology community (See, for example, [**4**]). In addition, we interpret the *epidemic threshold phenomenon* as a *transcritical bifurcation*.

The model Let $S(t)$, $I(t)$, and $R(t)$ be the number of individuals in the class of susceptible, infected, and removed at time t respectively. The SIR model assumes: (i) The rate at which susceptibles become infected is $\kappa S(t)I(t)$, where the transmission coefficient κ is a constant; (ii) the rate of transition from the infected class to the removed class is given by $\ell I(t)$, where the constant $1/\ell$ is the average length of the infectious period [**1, 2, 5, 7**]. With these assumptions, the differential equations describing the number of individuals in the three classes are

$$S'(t) = -\kappa S(t)I(t) \tag{1a}$$

$$I'(t) = \kappa S(t)I(t) - \ell I(t) \tag{1b}$$

$$R'(t) = \ell I(t) \tag{1c}$$

It is straightforward to verify that the total population $N = S(t) + I(t) + R(t)$ is a constant. Setting $N = 1$, the three variables become the fractions of the population in the respective classes. Typical initial conditions are $S_0 \approx 1$, $I_0 = 1 - S_0 \ll S_0$, and $R_0 = 0$. For more details on the SIR model, see [**5**] for a nice introduction.

Using an approximate solution of $R(t)$, Kermack and McKendrick [**1**] discovered the so-called *threshold phenomenon*: an epidemic occurs if and only if $S_0 > \ell/\kappa$. We illustrate this by showing two representative numerical solutions of $S(t)$, $I(t)$ and $R(t)$ in Figure 1. On the left with $S_0 < \ell/\kappa$, there is *no* epidemic; on the right with $S_0 > \ell/\kappa$, there *is* an epidemic. The parameter $\lambda \equiv \kappa/\ell$ is called the *reproduction rate*. It is the number of secondary infections produced by one primary infection in a totally susceptible population (see, for example, [**2**]).

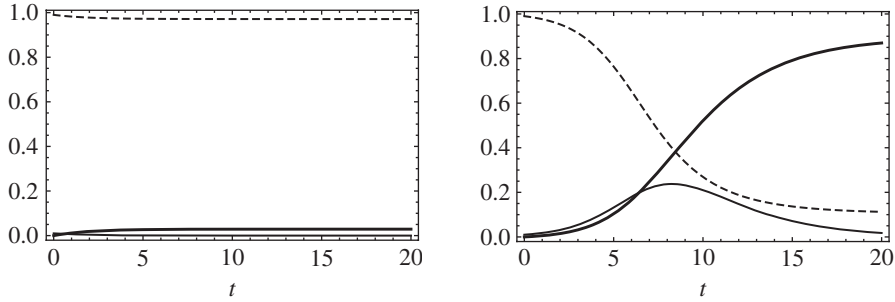

Figure 1. Two scenarios of the SIR model: $S(t)$, dashed line; $I(t)$, thin line; $R(t)$, thick line.

Asymptotic values We focus on $S(t)$, $I(t)$, and $R(t)$ as $t \to \infty$, and use them to visualize the epidemic threshold. Suppose $S(t)$, $I(t)$, and $R(t)$ eventually come to rest at some stable fixed point $(S_\infty, I_\infty, R_\infty)$. Then at this point the right hand sides of (1) must be zero. A quick way to realize that $I_\infty = 0$ is to use (1c). From (1a) and (1c), we can solve for S as a function of R: $S = S_0 e^{-\kappa R/\ell}$. Because $S + I + R = N \equiv 1$, (1c) can be rewritten as

$$R'(t) = \ell(1 - R - S_0 e^{-\kappa R/\ell}). \tag{2}$$

The asymptotic value R_∞ causes the right hand side to vanish, i.e.,

$$1 - R_\infty - S_0 e^{-\lambda R_\infty} = 0 \tag{3}$$

where $\lambda \equiv \kappa/\ell$. To solve (3) for R_∞, conventional wisdom suggests that a numerical or graphic method is needed, but the relatively unknown Lambert W function is useful.

The Lambert W function is an inverse of $T(w) = we^w$. In the real domain T has two inverse functions. We are concerned with the inverse $W : [-1/e, \infty) \to [-1, \infty)$, which has a minimum at $(-1, -1/e)$. For properties of W, consult [**3**]; for practical purposes, W can be invoked in *Maple* by `LambertW`, or *Mathematica* by `ProductLog`.

From the fact that $x = W(x)e^{W(x)}$, [**3**] derives the following property:

A solution for x in the equation $ax + b + ce^{dx} = 0$ (with $ad \neq 0$) is given by

$$x = -\frac{b}{a} - \frac{1}{d} W\left(\frac{cde^{-bd/a}}{a}\right)$$

as long as the domain constraints of W are satisfied.

Applying this to (3) yields

$$R_\infty = 1 + \frac{W(-S_0\lambda e^{-\lambda})}{\lambda}. \tag{4}$$

From $S_\infty + I_\infty + R_\infty = 1$ and $I_\infty = 0$, we get

$$S_\infty = 1 - R_\infty = -\frac{W(-S_0\lambda e^{-\lambda})}{\lambda}. \tag{5}$$

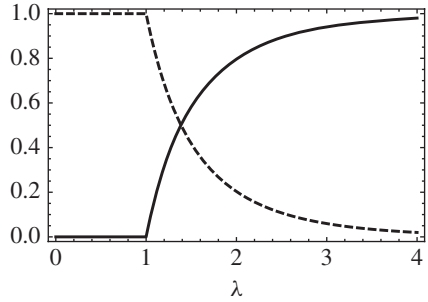

Figure 2. The epidemic threshold: asymptotic values S_∞ (dashed line) and R_∞ (solid line) have a sudden transition when λ exceeds one.

Plots of S_∞ and R_∞ as a function of λ are shown in Figure 2. The sudden increase of R_∞, the fraction of the population that has been affected by the epidemic, at $\lambda = 1$ is apparent. Algebraically, when $S_0 = 1$, the numerator of the second term in (4), $W(-\lambda e^{-\lambda})$, is simply $-\lambda$ for $\lambda \leq 1$, because W is the inverse function of $T(w) = we^w$ for $w \geq -1$. Therefore, $R_\infty = 0$ for $\lambda \leq 1$. Furthermore, R_∞ is strictly increasing for $\lambda > 1$ and tends to 1 as $\lambda \to \infty$.

Transcritical bifurcation The epidemic threshold can be viewed as a bifurcation point. A graphic analysis, using $S_0 = 1$, is given in Figure 3 (following [**6**]). The two components of R' according to (2), namely the line $y = 1 - R$ and the curve $y = e^{-\lambda R}$, are plotted. The dynamics of R are indicated along the R-axis: by an arrow to the right if $R' > 0$ and an arrow to the left if $R' < 0$

For $\lambda \neq 1$, the line and curve intersect twice, and one intersection is invariably $R = 0$. For $\lambda < 1$, the zero root is a stable fixed point; for $\lambda > 1$, the nonzero root is stable. As λ increases from less than 1 to greater than 1, the fixed points pass each other (at $R = 0$, of course) and exchange stability. Such behavior is called a *transcritical* bifurcation. (Note that "bifurcation" literally means a splitting into two.)

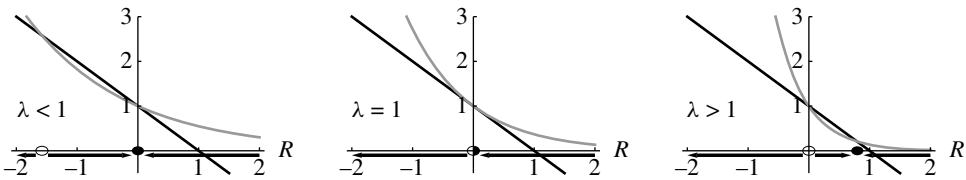

Figure 3. Two functions $1 - R$ and $e^{-\lambda R}$ intersect at fixed points (filled dot for stable, and hollow dot for unstable).

Conclusion The stable fixed point of the SIR model can be expressed in terms of the Lambert W function. The existence of an epidemic threshold is a phenomenon of transcritical bifurcation, for which the plot (Figure 2) of R_∞ as a function of λ is the bifurcation diagram.

Acknowledgment. I thank a referee for offering many suggestions for revising this note and pointing out [**4**].

Summary. The Lambert W function (an inverse of we^w) makes a surprise appearance in the analysis from a dynamical systems point-of-view of the SIR epidemiological model.

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. . . Three preconditions, three strokes of luck in the evolutionary arena, led to the scientific revolution. The first was the boundless curiosity and creative drive of the best minds. The second was the inborn power to abstract the essential qualities of the universe. . . . The third enabling precondition is what the physicist Eugene Wigner once called the unreasonable effectiveness of mathematics in the natural sciences. For reasons that remain elusive to scientists and philosophers alike, the correspondence of mathematical theory and experimental data in physics in particular is uncannily close. It is so close as to compel the belief that mathematics is in some deep sense the natural language of science. "The enormous usefulness of mathematics in the natural sciences," Wigner wrote, "is something bordering on the mysterious and there is no rational explanation for it. It is not at all natural that 'laws of nature' exist, much less that man is able to discover them. The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

—Edward O. Wilson, *Consilience*

The laws of physics are in fact so accurate as to transcend cultural differences. They boil down to mathematical formulae that cannot be given Chinese or Ethiopean or Maya nuances. Nor do they cut any slack for masculinist or feminist variations. We may even reasonably suppose that any advanced extraterrestrial civilizations, if they possess nuclear power and can launch spacecraft, have discovered the same laws, such that their physics could be translated isomorphically, point to point, set to point, and point to set, into human notation.

—Edward O. Wilson, *Consilience*