Example 5. Let

$$f(x) = \begin{Bmatrix} x^{-k}, & 0 < x \le 1, \\ x^{-1/k}, & x \ge 1, \end{Bmatrix} \quad \text{for } x \in (0, \infty) \text{ and arbitrary positive integer } k.$$

For further reading on related material, see S. Avital and S. Libeskind, "An Algebraic and Geometric Approach to Two Step Iterations of Bilinear Functions," *Amer. Math. Monthly* 91 (January 1984) 53–56; P. E. Conner and E. E. Floyd, *Differential Periodic Maps*, Springer-Verlag, New York, 1964; R. G. Kuller "On the Differential Equation $f' = f \circ g$, where $g \circ g = I$," *Math. Mag.* 42 (1969) 195–200; N. McShane, "On the Periodicity of Homeomorphism on Real Line," *Amer. Math. Monthly* 68 (1961); O. Shisha and C. B. Mehr, "On Involutions," *Journal of National Bureau of Standards* 71*B* (1967) 19–20.

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Involutions and Problems Involving Perimeter and Area

Joseph Wiener, Henjin Chi, and Hushang Poorkarimi, Pan American University
Edinburg, TX

William Parsons [CMJ 15 (November 1984) 429] asked which integer-sided right triangles have perimeter and area equal. Here we illustrate how properties of involutions can be used to answer this and similar questions.

First, consider a triangle with legs x and y. Setting its perimeter and area equal is equivalent to requiring that

$$y = \frac{4x - 8}{x - 4} \qquad (x \neq 4). \tag{1}$$

For a rectangle of dimensions x and y, the equality of its perimeter and area is equivalent to

$$y = \frac{2x}{x - 2} \qquad (x \neq 2). \tag{2}$$

The bilinear functions (1), (2) are, respectively, special cases of (2), (4) in the above capsule. Thus, $f(x) = (4x - 8)/(x - 4)$ and $g(x) = 2(x)/(x - 2)$ are involutions.

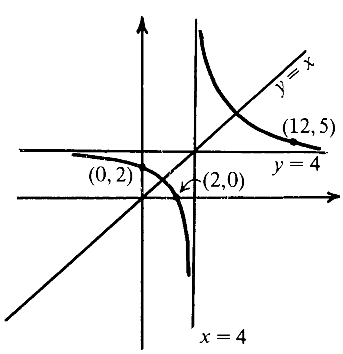

Figure 1. $y = \dfrac{4x - 8}{x - 4}$.

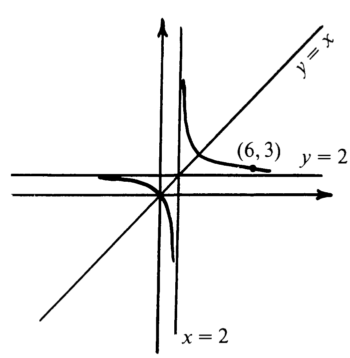

Figure 2. $y = \dfrac{2x}{x - 2}$.

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As noted in the previous classroom capsule, the graph of every involution $F(x)$ is symmetric with respect to the line $y = x$. Furthermore, every involution is 1-1 since $F(x_1) = F(x_2)$ implies that $x_1 = F\{F(x_1)\} = F\{F(x_2)\} = x_2$. It is also easily verified (with the exception of $F(x) = x$) that $F(x) = (ax + b)/(cx - a)$ is decreasing if and only if $a^2 + bc > 0$.

As Figure 1 illustrates, we need only consider (1) for $4 < x \leq 12$. From

$$y = \frac{4x - 8}{x - 4} = 4 + \frac{8}{x - 4},$$

we note that $x - 4$ divides 8. Thus, checking $x - 4 = 2^k$ ($k = 0, 1, 2, 3$), we obtain integer solutions $(5, 12)$ and $(6, 8)$. By symmetry of the involution's graph about the line $y = x$, we also have solutions $(12, 5)$ and $(8, 6)$. Thus, $(5, 12)$ and $(6, 8)$ are the two solutions that describe all integer-sided right triangles having perimeter and area equal. In similar fashion, Figure 2 shows that it suffices to consider $2 < x \leq 6$. Since

$$y = \frac{2x}{x - 2} = 2 + \frac{4}{x - 2}$$

requires that $x - 2$ divide 4, we readily obtain solutions $(3, 6)$ and $(4, 4)$, and (by symmetry) the solution $(6, 3)$. Thus, there exist precisely two integer-sided rectangles that have perimeter and area equal.

In the above mentioned capsule, Parsons also formulated the following conjecture (a proof of which he cited in *School Science and Mathematics* 76 (1976) 83–84):

> *For every natural number n, there is at least one primitive Pythagorean triangle whose area equals n times its perimeter.*

This is equivalent to the existence of at least one solution to

$$n\left(x + y + \sqrt{x^2 + y^2}\right) = \frac{xy}{2}, \tag{3}$$

where the integers x and y are relatively prime. From (3) we obtain the involution

$$y = \frac{4nx - 8n^2}{x - 4n} = 4n + \frac{8n^2}{x - 4n}. \tag{4}$$

The graph of (4) shows that we need only consider $4n < x \leq 8n^2 + 4n$. Since $x - 4n$ must divide $8n^2$, we need to consider all possibilities $x - 4n = 2^k n^j$ ($k = 0, 1, 2, 3$ and $j = 0, 1, 2$). Of course, it is a simple matter to write a computer program which, for each given natural number n, will output all integral solutions (x, y) of (4). But do we know if solutions actually exist for each n? The answer is yes.

Theorem. *For each natural number n, there exists a primitive Pythagorean triangle with legs*

$$x = 4n + 1 \quad \textit{and} \quad y = 8n^2 + 4n \tag{5}$$

in which the area equals n times the perimeter. If $n \geq 3$ is odd, there exists at least one other such primitive Pythagorean triangle with legs

$$x = n^2 + 4n \quad \textit{and} \quad y = 4n + 8. \tag{6}$$

Proof. First note that

$$(4n + 1)^2 + (8n^2 + 4n)^2 = (8n^2 + 4n + 1)^2$$

and

$$(n^2+4n)^2+(4n+8)^2=(n^2+4n+8)^2,$$

so that x, y and $\sqrt{x^2+y^2}$ form a right triangle for each natural number n. To prove that x, y in (5) are relatively prime, assume that they have a common factor p. Then

$$4n+1=ip \quad \text{and} \quad 8n^2+4n=jp$$

for integers i, j. Multiplying the first equation by $2n$ and subtracting from the second, we have $2n=mp$ (m, integral). From this and the first equation, we get $kp=1$ (k, integral) and hence $p=1$. Finally, we verify that x, y in (6) are relatively prime when n is odd. Assume that

$$n^2+4n=ip \quad \text{and} \quad 4n+8=jp$$

for integers i, j. Since n^2+4n is odd, p must be odd and j must be a multiple of 4. Hence, $n+2=mp$ (integral m). Subtracting $n^2+2n=mnp$ from $n^2+4n=ip$, we get $2n=kp$ (integral k). Since p must divide n, assume that $n=pq$ for some integer q. Then, subtracting $n=pq$ from $n+2=mp$, we obtain $2=sp$ (integral s). Hence, $p=1$. This completes the proof of our theorem.

More on the Series for ln 2

Leonard Gillman, The University of Texas at Austin, Austin, TX

Norman Schaumberger [CMJ 18 (May 1987) 223–225] derived the series

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \tag{1}$$

directly from the inequality

$$\left(1+\frac{1}{k}\right)^k<e<\left(1+\frac{1}{k}\right)^{1+k} \tag{2}$$

and extended the method to obtain series for $\ln n$ for all n.

Here is a variant procedure. In place of (2), our point of departure is the existence of Euler's constant. For completeness, we first give a simple geometric derivation of this constant; the only sophisticated step is that a bounded monotone sequence converges.

We define

$$H_n=1+\frac{1}{2}+\cdots+\frac{1}{n} \tag{3}$$

and

$$\gamma_n=H_n-\ln n. \tag{4}$$

It is also convenient to put

$$\gamma_n'=H_{n-1}-\ln n. \tag{5}$$

Thus,

$$\gamma_n=\gamma_n'+\frac{1}{n}. \tag{6}$$