x to determine how many piles to put above the selected pile in each pass. Then you can do the trick the other way, with piles of three followed by piles of five—write $p = 5x + y$ with $0 \le x \le 2$ and $0 \le y \le 4$. Dickson [**3**] and Onnen[**9**] wrote about this generalization.

When you've mastered it, move on to a mixed radix with three-digit numbers. But if you are serious about doing mixed radix Gergonne tricks, it pays to learn the right to left digit algorithm for expressing numbers in the strange bases you invent.

REFERENCES

1. A. Bogomolny, The computer as a magician, `http://www.maa.org/editorial/knot/ComputerAsMagician.html`.
2. David Britland, Cardopolis, `http://cardopolis.blogspot.com/2008_01_20_archive.html`.
3. L. E. Dickson, Gergonne's pile problem, *Bull. Amer. Math. Soc.* **1** (1895) 184–186, `http://projecteuclid.org/euclid.bams/1183414376`.
4. M. Gardner, *Mathematics, Magic and Mystery*, Dover, 1956.
5. J. D. Gergonne, Récréations mathématiques. Recherches sur un tour de cartes, *Annales de Mathématiques* **iv** (1813–1814) 276–283, `http://archive.numdam.org/item?id=AMPA_1813-1814__4__276_1`.
6. J. Harrison, T. Brennan, and S. Gapinski, The Gergonne p-pile problem and the dynamics of the function $x \to \lfloor (x + r)/p \rfloor$, *Discrete Applied Mathematics* **82** (1998) 103–113. doi:10.1016/S0166-218X(97)00132-7
7. D. Knuth, *The Art of Computer Programming, Volume 3: Sorting and Searching*, 3rd ed., Addison-Wesley, 1997.
8. Max Maven, in D. Wolfe and T. Rodgers, eds., *Puzzlers' Tribute: A Feast for the Mind*, AK Peters, 2001.
9. H. Onnen, Gergonne's pile problem, *Bull. Amer. Math. Soc.* **16** (1909) 121–130. `http://projecteuclid.org/euclid.bams/1183420544`. doi:10.1090/S0002-9904-1909-01874-9
10. Eric Shrader and Keith Riding, Cut the knot, `http://www.cut-the-knot.org/arithmetic/rapid/CardTrick.shtml`.
11. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 13th ed., Dover, 1987.

Summary Gergonne's three pile card trick has been a favorite of mathematicians for nearly two centuries. This new exposition uses the radix sorting algorithm well known to computer scientists to explain why the trick works, and to explore generalizations. The presentation suggests strategies for introducing the trick and base three arithmetic to elementary school students.

A GM-AM Ratio

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Let $\mathrm{GM}(a_1, \ldots, a_n)$ and $\mathrm{AM}(a_1, \ldots, a_n)$ denote the geometric and arithmetic mean of positive real numbers $a_1, \ldots, a_n$, respectively. Kubelka [**1**] proves that for any $s > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{GM}(1^s, \ldots, n^s)}{\mathrm{AM}(1^s, \ldots, n^s)} = \frac{s+1}{e^s}. \tag{1}$$

He uses the squeeze theorem and a Riemann sum argument. In pursuing a simpler method to show (1), I realized that $\ln[\mathrm{GM}(1^s, \ldots, n^s)/n^s]$ is a Riemann sum

Math. Mag. **83** (2010) 49–50. doi:10.4169/002557010X479992.

for the area between the x-axis and $y = \ln(x^s)$ from $x = 0$ to $x = 1$ and that $\mathrm{AM}(1^s, \dots, n^s)/n^s$ is also a Riemann sum for the area between the x-axis and $y = x^s$ from $x = 0$ to $x = 1$. These observations yield the following two stronger results: For any real number $s > -1$,

$$\lim_{n\to\infty} \frac{\mathrm{GM}(1^s, \dots, n^s)}{n^s} = e^{-s}, \tag{2}$$

$$\lim_{n\to\infty} \frac{\mathrm{AM}(1^s, \dots, n^s)}{n^s} = \frac{1}{s+1}. \tag{3}$$

To show (2), it is equivalent to show that

$$\lim_{n\to\infty} \Big[\ln[\mathrm{GM}(1^s, \dots, n^s)] - \ln(n^s) \Big] = -s.$$

In fact, using the Riemann sum yields that

$$\lim_{n\to\infty} \Big[\ln[\mathrm{GM}(1^s, \dots, n^s)] - \ln(n^s) \Big] = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left(\frac{i}{n}\right)^s = \int_0^1 \ln(x^s)\,dx = -s.$$

Similarly, (3) follows from

$$\lim_{n\to\infty} \frac{\mathrm{AM}(1^s, \dots, n^s)}{n^s} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^s = \int_0^1 x^s\,dx = \frac{1}{s+1}.$$

Now (1) follows from (2) and (3) by division. This method works for $s > -1$, while Kubelka [**1**] assumes $s > 0$.

Some other limits that involve $\mathrm{GM}(1^s, \dots, n^s)$ and $\mathrm{AM}(1^s, \dots, n^s)$, and that do not follow directly from (1), can be evaluated by using (2) and (3). For example, using (2) and (3) we can show that

$$\lim_{n\to\infty} \frac{[\mathrm{GM}(1^s, \dots, n^s)]^2}{[\mathrm{AM}(1^s, \dots, n^s)]^2 + n^s\ \mathrm{GM}(1^s, \dots, n^s)} = \frac{e^{-s}(s+1)^2}{e^s + (s+1)^2}.$$

REFERENCE

1. R. P. Kubelka, Means to an end, *Mathematics Magazine* **74** (2001) 141–142.

Summary If $s > -1$, then the limit of the ratio of the geometric mean of $1^s, \dots, 2^s$ to their arithmetic mean, as n increases to infinity, is $(s+1)/e^s$. This is proved using Riemann sums. Similar limits are also established for the arithmetic and geometric means separately.