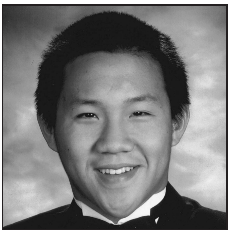
A Characterization of a Quadratic Function in $\mathbb{R}^n$

Conway Xu

doi:10.4169/074683410X488692



Conway Xu (conway.xu@gmail.com) is currently a senior in the Science, Mathematics, and Computer Science Magnet Program at Montgomery Blair High School in Silver Spring, MD. He's interested in mathematics and science and would like to thank his math teacher and coach, Eric Walstein, for teaching him multivariable calculus and differential equations.

For a differentiable function $f(x)$ on $x \in (-\infty, \infty)$, Stenlund [**2**] in this JOURNAL obtained an elegant characterization: $f(x)$ is a quadratic function, i.e., $f(x) = ax^2 + bx + c$ $(a \neq 0)$, if and only if the intersection of any two tangent lines lies midway horizontally between the tangent points. Krasopoulos [**1**], in this JOURNAL, generalized this result to n-dimensional space $\mathbb{R}^n$, but only partially. Specifically, he proved that if a function $f(\mathbf{x})$ is defined by

$$f(\mathbf{x}) = \mathbf{x}^T Q\,\mathbf{x} + \mathbf{c}^T\mathbf{x} + d, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where Q is a real $n \times n$ symmetric non-zero matrix, $\mathbf{c}$ is a vector and d is a constant, then the midpoint of the line segment $\overline{\mathbf{x_1x_2}}$ lies in the intersection of the tangent planes at $\mathbf{x}_1$ and $\mathbf{x_2}$. He didn't address the converse:

Theorem. *If the intersection of the tangent planes to the graph of $y = f(\mathbf{x})$, where f is a differentiable function defined everywhere on $\mathbb{R}^n$, at points $\mathbf{x}_1$ and $\mathbf{x}_2$ always contains the midpoint of $\overline{\mathbf{x_1x_2}}$, then f has the form (1).*

In this note I prove this result. Note that if the midpoint $\bar{\mathbf{x}} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ of the line segment $\overline{\mathbf{x_1x_2}}$ belongs to the intersection of the tangent planes at $\mathbf{x}_1$ and $\mathbf{x_2}$, then

$$f(\mathbf{x}_1) + (\bar{\mathbf{x}} - \mathbf{x}_1)^T \nabla f(\mathbf{x}_1) = f(\mathbf{x}_2) + (\bar{\mathbf{x}} - \mathbf{x}_2)^T \nabla f(\mathbf{x}_2), \tag{2}$$

where $\nabla f(\mathbf{x}) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)^T$ denotes the gradient of $f(\mathbf{x})$.

Now consider the function $g(\lambda)$ defined by $g(\lambda) = f(\lambda\mathbf{x})$, where $\lambda \in (-\infty, \infty)$ and $\mathbf{x} \in \mathbb{R}^n$ is considered a parameter. Then $g'(\lambda) = \mathbf{x}^T \nabla f(\lambda\mathbf{x})$ and the two tangent lines at λ_1 and λ_2 are given by

$$T_1(\lambda) = g'(\lambda_1)(\lambda - \lambda_1) + g(\lambda_1),$$

and

$$T_2(\lambda) = g'(\lambda_2)(\lambda - \lambda_2) + g(\lambda_2),$$

respectively. Letting $\mathbf{x}_1 = \lambda_1\mathbf{x}$ and $\mathbf{x}_2 = \lambda_2\mathbf{x}$ in (2) yields $T_1(\bar{\lambda}) = T_2(\bar{\lambda})$, where $\bar{\lambda} = (\lambda_1 + \lambda_2)/2$. That is, the intersection of any two tangent lines lies midway horizontally between the tangent points. Then applying Stenlund's result from [**2**] yields that $g(\lambda)$ must be a quadratic function, that is,

$$g(\lambda) = f(\lambda \mathbf{x}) = A(\mathbf{x})\lambda^2 + B(\mathbf{x})\lambda + C(\mathbf{x}), \tag{3}$$

which implies that

$$\begin{aligned} g'(\lambda) &= \mathbf{x}^T \nabla f(\lambda \mathbf{x}) = 2A(\mathbf{x})\lambda + B(\mathbf{x}), \\ g''(\lambda) &= \mathbf{x}^T \nabla^2 f(\lambda \mathbf{x})\, \mathbf{x} = 2A(\mathbf{x}), \end{aligned} \tag{4}$$

where $\nabla^2 f(\mathbf{x})$ is the $n \times n$ matrix defined by $\nabla^2 f(\mathbf{x}) = \left(\partial^2 f/\partial x_i \partial x_j\right)$. Letting $\lambda = 0$ in (4) yields

$$\begin{aligned} B(\mathbf{x}) &= \mathbf{x}^T \nabla f(\mathbf{0}), \\ 2A(\mathbf{x}) &= \mathbf{x}^T \nabla^2 f(\mathbf{0})\, \mathbf{x}. \end{aligned} \tag{5}$$

The combination of (5) and (3) with $\lambda = 1$ leads to the conclusion that

$$f(\mathbf{x}) = \mathbf{x}^T Q\, \mathbf{x} + \mathbf{c}^T \mathbf{x} + d,$$

where $Q = \nabla^2 f(\mathbf{0})/2$, $\mathbf{c} = \nabla f(\mathbf{0})$, and $d = f(\mathbf{0})$. That is, the function $f(\mathbf{x})$ has the desired form.

Acknowledgments. I would like to thank the two anonymous reviewers for their helpful comments and suggestions, which led to a substantial improvement of this note.

Summary. It is proved that a scalar-valued function $f(\mathbf{x})$ defined in n-dimensional space must be quadratic, if the intersection of tangent planes at $\mathbf{x}_1$ and $\mathbf{x}_2$ always contains the midpoint of the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$. This is the converse of a result of Krasopoulos proved in the *College Mathmatics Journal* in 2003.

References

1. P. T. Krasopoulos, Tangent planes of a quadratic function, *College Math. J.* **34** (2003) 205–206. doi:10.2307/3595802
2. M. Stenlund, On the tangent lines of a parabola, *College Math. J.* **32** (2001) 194–196. doi:10.2307/2687469

"I hope you slept well," he said.

"Yes, isn't it lovely?" Jenny replied, giving two rapid little nods. "But we had such awful thunderstorms last week." (Jenny is deaf.)

Parallel straight lines, Denis reflected, meet only at infinity. He might talk for ever of care-charmer sleep and she of meteorology till the end of time. Did one ever establish contact with anyone? We are all parallel straight lines, Jenny was only a little more parallel than most.

—from *Chrome Yellow* by Aldous Huxley