The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof

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One of the most simple and successful techniques for solving systems of linear equations is to reduce the coefficient matrix of the system to **reduced row echelon form**. This is accomplished by applying a sequence of elementary row operations (see, e.g., [**1**, p. 5]) to the coefficient matrix until a matrix B is obtained which satisfies the following description:

> If a row of B does not consist entirely of zeros then the first nonzero number in the row is a 1 (usually called a leading 1).
>
> If there are any rows that consist entirely of zeros, they are grouped together at the bottom of B.
>
> In any two successive non-zero rows of B, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
>
> Each column of B that contains a leading 1 has zeros everywhere else.

The matrix B is said to be in reduced row echelon form.

It is well known that if A is an $m \times n$ matrix and x is an $n \times 1$ vector, then the systems $Ax = 0$ and $Bx = 0$ have the same solution set. However, the solution to the system $Bx = 0$ may be read off immediately from the matrix B. For example, consider the system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + 3x_2 + 7x_3 &= 0 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \end{aligned}$$

The matrix of coefficients is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{pmatrix}$$

and through use of elementary row operations we obtain

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is seen to be in reduced row echelon form. Column 3 (unlike columns 1 and 2) does not contain a leading 1. We call such a column a **free** column, because in the solution to the system $Bx = 0$ (and hence $Ax = 0$), the variable corresponding to that column is a free parameter. Thus we can set $x_3 = t$, and read off the equations $x_1 + 2t = 0$ (row 1) and $x_2 + t = 0$ (row 2). The solution set is then $x_1 = -2t$, $x_2 = -t$, $x_3 = t$.

An important theoretical result is that the reduced row echelon form of a matrix is unique. Most texts either omit this result entirely or give a proof which is long and very technical (see [**2**, p. 56]). The following proof is somewhat clearer and less complicated than the standard proofs.

THEOREM. *The reduced row echelon form of a matrix is unique.*

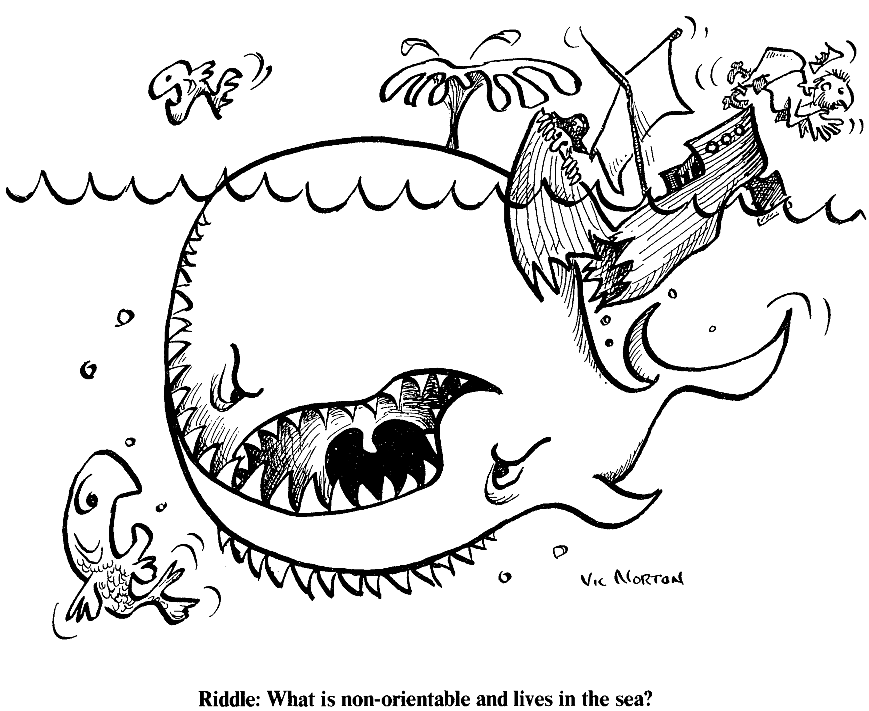
Proof. Let A be an $m \times n$ matrix. We will proceed by induction on n. For $n = 1$ the proof is obvious. Now suppose that $n > 1$. Let A' be the matrix obtained from A by deleting the nth column. We observe that any sequence of elementary row operations which places A in reduced row echelon form also places A' in reduced row echelon form. Thus by induction, if B and C are reduced row echelon forms of A, they can differ in the nth column only. Assume $B \neq C$. Then there is an integer j such that the jth row of B is not equal to the jth row of C. Let u be any column vector such that $Bu = 0$. Then $Cu = 0$ and hence $(B - C)u = 0$. We observe that the first $n - 1$ columns of $B - C$ are zero columns. Thus the jth coordinate of $(B - C)u$ is $(b_{jn} - c_{jn})u_n$. Since $b_{jn} \neq c_{jn}$ we must have $u_n = 0$. Thus any solution to $Bx = 0$ or $Cx = 0$ must have $x_n = 0$. It follows that both the nth columns of B and C must contain leading 1's, for otherwise those columns would be free columns and we could arbitrarily choose the value of x_n. But since the first $n - 1$ columns of B and C are identical, the row in which this leading 1 must appear must be the same for both B and C, namely the row which is the first zero row of the reduced row echelon form of A'. Because the remaining entries in the nth columns of B and C must all be zero, we have $B = C$, which is a contradiction. This establishes the theorem.

We remark that this proof easily generalizes to the following proposition: *Let A be an $m \times n$ matrix with row space W. Then there is a unique $m \times n$ matrix B in reduced row echelon form such that the row space of B is W.* (For another proof, see [**2**, p. 56].)

References

[1] H. Anton, Elementary Linear Algebra, Wiley, New York, 1977.

[2] K. Hoffman and R. Kunze, Linear Algebra, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.



Riddle: What is non-orientable and lives in the sea?

—Robert Messer

(See News and Letters if you give up.—ed.)