Cars, Goats, π, and e

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People seldom use information as effectively as possible when faced with decisions under uncertain conditions; a famous example is the Monty Hall Problem [**3**]. We examine two particular variations on the problem, derived respectively from papers [**7**] and [**6**], and we show a curious connection between the probability of winning and the numbers π and Euler's number e. That is, players can approximate π and e by playing the variation games repeatedly using the best strategies, in a way very similar to approximating π by dropping Buffon's needle [**1**].

The Monty Hall Problem The problem gets its name from a TV game show, "Let's Make A Deal," hosted by Monty Hall. We summarize as follows:

PROBLEM 1. *There are 3 closed doors: One hides a car and each other one a goat. The game's host knows what is behind each door, but the contestant does not and tries to win the car by choosing the door that hides it. The host asks the contestant to choose a door; then, the host opens a door drawing it randomly from the remaining ones without the car, and asks the player if he/she would like to exchange the chosen door with the still-closed one.*

Is it better for the contestant to change the door or to keep the first he chose?

As detailed in an article in this issue of the MAGAZINE [**5**], the wrong solution is quite common, with people perhaps reasoning as follows: "There are two closed doors left, therefore the probability that the car is behind either one is $1/2$; so there is no reason to change." Of course, the correct choice is always to change. In fact, if the contestant changes, he/she loses if and only if the first chosen door hides the car, and therefore the probability of winning is $2/3$.

Monty Hall and π Let's examine an extension of Monty Hall's problem, originally described in a paper in the Italian journal *Archimede* [**7**]. In this version there are n doors ($n \geq 3$ and n odd), which coincides with the original if $n = 3$. As in Problem 1, only one door conceals a car while each of the others hides a goat. The game's host knows where the car is, but the contestant does not and must guess it to win. The game evolves as follows:

PROBLEM 2. *There are n closed doors, n is odd, and $n \geq 3$.*

1. *The contestant picks a door from among the closed and not yet chosen ones.*
2. *If no closed and not yet chosen door exists, then the game is over and the last chosen door is opened. Otherwise, the game's host opens a door, picking it randomly from among the doors that are closed, not yet chosen, and without the car. Then he asks the contestant if he/she wants to switch to one of the closed and not yet chosen doors.*
3. *If the contestant decides not to change, the last chosen door is opened and the game is over. Otherwise, the game goes back to Step 1.*

What is the best strategy for the contestant?

It is supposed that the number n of doors is odd so that, after the host opens a door, the contestant still has the opportunity to change his last choice. We will now show what the optimal strategy is, and the corresponding probability to win. For convenience, we refer to the probability that a particular door conceals the car as "the probability of the door."

PROPOSITION 1. *In Problem 2, the best strategy for the contestant is to change doors at every opportunity. In this case the probability of winning is*

$$p(n) = \frac{2 \cdot 4 \cdots (n-3)(n-1)}{3 \cdot 5 \cdots (n-2)n}$$

and

$$\lim_{n\to\infty} p(n)^2 \cdot n = \frac{\pi}{2}. \tag{1}$$

Proof. We name the probability of winning as $p(n)$, assuming that the contestant always decides to change. The probability that the first chosen door hides the car is $\frac{1}{n}$. After the host has opened the first door without the car, the probability of each of the $n-2$ unchosen and still closed doors is $\frac{n-1}{n-2}\frac{1}{n}$, because all $n-1$ of the probabilities must add to one. If n were 3, we would be done, concluding that the player who switches wins with probability 2/3. In the general case, it is as if the player has begun a new game with $n-2$ doors, but with the chance of winning increased by a factor of $(n-1)/n$. Thus, $p(n) = \frac{n-1}{n} p(n-2)$, leading to the product formula given. This makes it easy to prove by induction that $p(n) > 1/n$ for all odd $n \geq 3$. Changing is the best strategy, because if the contestant does not switch (Step 3), then the probability of getting the prize will, for some k, be

$$\frac{(n-1)(n-3)\cdots(n-2k-1)}{n(n-2)\cdots(n-2k)} \frac{1}{(n-2k-2)}$$
$$< \frac{(n-1)(n-3)\cdots(n-2k-1)}{n(n-2)\cdots(n-2k)} p(n-2k-2) = p(n).$$

From the Wallis formula $\lim_{n\to\infty} \left(\frac{2\cdot4\cdots(n-3)(n-1)}{3\cdot5\cdots(n-4)(n-2)}\right)^2 / n = \pi/2$ [**2**], we deduce the desired result (1). ■

Monty Hall and *e* Our next version of the Monty Hall Problem with n doors is similar to one described in the *Monthly* [**6**]. The doors are replaced by boxes, because we need to shuffle them. Moreover, $n \geq 3$ and n is not necessarily odd. Only one box hides a prize, while all the others are empty. The host knows where the prize is, but the contestant does not and must guess it to win. The new version of the problem is:

PROBLEM 3. *There are n closed boxes, where $n \geq 3$. The contestant picks a closed box. If exactly one other closed box remains, the game is over and the last chosen box is opened to see whether the contestant wins the prize. Otherwise:*

1. *The host opens a box at random from among the empty, unchosen, and closed ones.*
2. *Then the host secretly shuffles the closed boxes, except the one just chosen by the contestant.*
3. *Finally, the host asks if the contestant wants to change boxes. If the contestant switches, the game continues from Step 1; otherwise the last chosen box is opened and the game is over.*

As in Problems 1 and 2, the question is to find the optimal strategy for the contestant.

The next proposition proves that, as in Problem 2, the best strategy is always to agree to change. Surprisingly, the corresponding probability of winning involves Euler's number e.

PROPOSITION 2. *In Problem 3, the best strategy for the contestant is always to change the box. In this case the probability of winning is*

$$p(n) = \sum_{i=1}^{n-2} \frac{(-1)^{i-1}}{i!} + \frac{(-1)^n}{(n-2)!n} \tag{2}$$

and

$$\lim_{n\to\infty} p(n) = 1 - \frac{1}{e}. \tag{3}$$

Proof. If the game returns to Step 1 with k closed boxes for the contestant to choose from, let p_k denote the probability of each box. Hence $p_n = \frac{1}{n}$ is the probability of the first chosen box. As before, after the host has opened the first box with no prize, each closed box except that chosen has probability $p_{n-2} = \frac{n-1}{n-2}\frac{1}{n} > \frac{1}{n} = p_n$. Therefore a shrewd contestant changes boxes. (We record for future reference that $p_{n-2} < \frac{1}{n-2}$.) Then, if $n > 3$, the host opens another empty box, randomly chosen from the $n-3$ available, and shuffles the remaining closed boxes except for the last chosen. Thus, each unchosen and closed box has probability

$$p_{n-3} = \frac{1-p_{n-2}}{n-3} = \frac{1}{n-3} - \frac{1}{(n-3)(n-2)} + \frac{1}{(n-3)(n-2)n} > p_{n-2}.$$

The last inequality is true since

$$p_{n-2} < \frac{1}{n-2} \text{ implies } \frac{1}{n-2} < \frac{1-p_{n-2}}{n-3} = p_{n-3},$$

which shows in turn that $p_{n-3} < \frac{1}{n-3}$. Assuming that $p_k = \frac{1-p_{k+1}}{k}$, by reverse induction we have $\frac{1}{k+1} < p_k < \frac{1}{k}$ and

$$p_k = (k-1)!\left[\sum_{i=k}^{n-2} \frac{(-1)^{i-k}}{i!} + \frac{(-1)^{n-k+1}}{(n-2)!n}\right] \quad \text{if } 1 \le k \le n-3.$$

Therefore, $p_n = \frac{1}{n} < p_{n-2} < \frac{1}{n-2} < \cdots < \frac{1}{k+1} < p_k < \frac{1}{k} < \cdots < p_1$ and the best strategy is to always change the box. We conclude (2) by setting $k = 1$, and (3) follows from the well-known Maclaurin series. ∎

In an article by Lucas and Rosenhouse [**4**], we learned of another extension of the Monty Hall Problem where the probability of winning is $1 - \frac{1}{e}$. Their paper identifies even more phenomena with this seemingly ubiquitous probability.

Final remarks It is interesting to observe that, while in the game of Problem 2 $p(n) \approx \sqrt{\frac{\pi}{2}}/\sqrt{n} \to 0$ as $n \to \infty$, in the game of Problem 3 $\lim_{n\to\infty} p(n) = 1 - \frac{1}{e} > 0.632$ is positive. In fact, in contrast to Problem 2, it is no longer true that if at some time the contestant chooses the box with the prize and then moves to another box, then the contestant is doomed to lose. So, this is a much better game for the contestant.

As observed above, it is possible to approximate π and e playing the games from Problems 2 and 3 over and over again. Play with as many doors as practical, always change doors, and record the frequencies of victories. As you may suspect, the formula $2np(n)^2$, derived from Proposition 1, is not the best way to approximate π; for example, it gives $\pi \approx 3.13$ for $n = 101$. The formula $\frac{1}{1-p(n)}$ for e does much better: if $n = 10$, it gives correctly the first six decimal digits of e.

Finally we notice that, while in Buffon's needle problem the appearance of π is not unexpected (π is the average of the possible inclinations of the needle), a geometrical interpretation of Proposition 1 is not evident.

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A Graph Theoretic Summation of the Cubes of the First *n* Integers

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The complete graph K_{n+1} has $n + 1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph K_{n+1}, introducing vertices one at a time, and counting new edges incident to each new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$ [**1**].

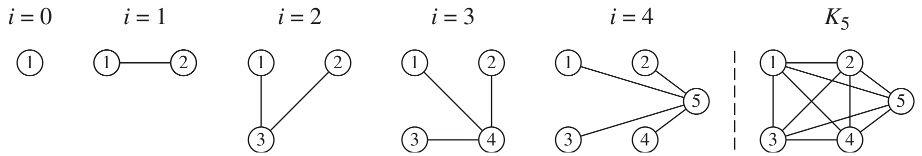

Figure 1 $\sum_{i=1}^{4} i = \binom{4+1}{2}$

Since $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$ it seems natural to look for a combinatorial proof that also uses graphs. The relevant graphs turn out to be *bipartite*, meaning that the vertices are partitioned into two sets and edges occur only between vertices from different parts.