CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

A Non-Visual Counterexample in Elementary Geometry

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Future math teachers, whether at the primary or secondary level, should enter their profession with a solid foundation in geometry. There are times when, in teaching elementary topics to such students, we have an opportunity to introduce more profound ideas at the same time. An example of this occurs in the context of area and perimeter. Every student needs to understand that there is no dependence whatsoever between a change in perimeter and a change in area—one can go up or down while the other remains constant. (Some theoretical background on this topic may be found in Courant and Robbins [**1**, Ch. VII], and in a more general discussion leading to the general isoperimetric problem and its dual in Jacobs [**2**, pp. 307–328].)

In my experience, many students do not comprehend this as it contradicts their erroneous intuition that an increase in one of these quantities leads to an increase in the other. An exchange similar to the following takes place between me (Instructor) and my students nearly every year.

Instructor: If two polygons are congruent, obviously they must have the same area and the same perimeter. Is the opposite true? That is, if two polygons have the same area and the same perimeter, do they have to be congruent?

Students (after some work): No, there is quite a simple example. Take a trapezoid with a right angle and attach a right triangle in two different ways, like this:

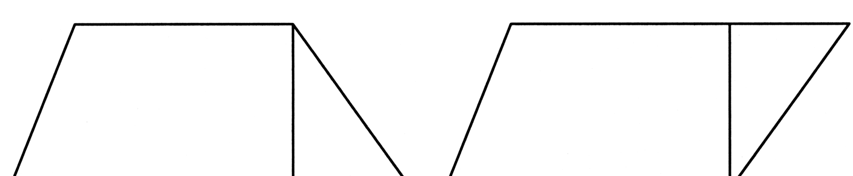

Figure 1.

Instructor: Very good, so it's not the case for quadrilaterals. But what about triangles? As you know, there are some ways in which triangles are special polygons.

For instance, if one triangle has all three sides the same length as another triangle, then the triangles must be congruent; but this isn't true for any polygons with more than three sides. So maybe, if two *triangles* have the same perimeter and the same area, they must be congruent.

Having found the trapezoid-parallelogram pair quite easily, the class discovers that it is surprisingly difficult to come up with a counterexample for triangles, so conjectures arise that for triangles this may be true. There are, in fact, a number of ways of showing that there are non-congruent triangles with equal areas and perimeters, but we do not know of any really simple constructions.

Some arguments use algebraic techniques in conjunction with results such as the Pythagorean theorem or Heron's formula. For example, starting with the 3-4-5 right triangle (so that at least some of the numbers are nice), we might ask whether there is an isosceles (again, to keep things nice) triangle with a perimeter of 12 and an area of 6, say with sides x, x, and $12 - 2x$. If so, then Heron squared results in the cubic equation $x^3 - 15x^2 + 72x - 111 = 0$, which has a root close to 3.467, and thus there is such an example.

Similarly, if we let x denote the base of this triangle, then the altitude is $12/x$, and therefore, using the Pythagorean theorem we find an equation for the perimeter:

$$x + 2\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{12}{x}\right)^2} = 12.$$

When simplified, this also results in a cubic equation, and again we get a solution. There are of course more variations of the same idea.

However, many students have weak backgrounds in algebra and are uncomfortable with cubic equations and even the quadratic formula, and hence there are pedagogical advantages to geometric solutions. More importantly, though, such solutions may provide insights beyond the specific problem. For many, this is their first encounter with the construction of a non-visual counterexample in elementary geometry, of a proof of existence (without the actual presentation of the object shown to exist). We present two approaches here, with variations.

Fixed perimeter solutions. Let $\triangle ABC$ be an equilateral triangle of side a, and form a new triangle $\triangle A'B'C'$ with the same perimeter by reducing the base by 2ε and increasing the other two sides by ε each. Clearly, the two triangles are not congruent since the second one is not equilateral. Let their areas be S and S' respectively. If they are equal, then we have found two non-congruent triangles with equal perimeters and equal areas. If they are not equal, then either $S > S'$ or $S' > S$. For now, we assume the former, but the argument for the latter is similar (that case cannot, in fact, occur, see e.g. [**2**]).

Imagine a piece of string of length $2a$ having its ends fixed at points B and C, and a pencil inside the string at point A so that the string is taut. Suppose that the pencil is moved from A towards the line of BC with the string remaining fully stretched (this of course forms a portion of an ellipse, but it is not necessary to mention this to the students). Each point on this curve is the third vertex of a triangle whose perimeter is clearly $3a$. As to the area, it decreases continually from S, the area of the equilateral triangle, to 0, as the pencil gets to the line containing BC. By the continuity principle (which is referred to only at an intuitive level at this stage), the curve necessarily contains the third vertex C'' of a triangle ABC'' whose area is S' but is clearly not congruent to $\triangle A'B'C'$.

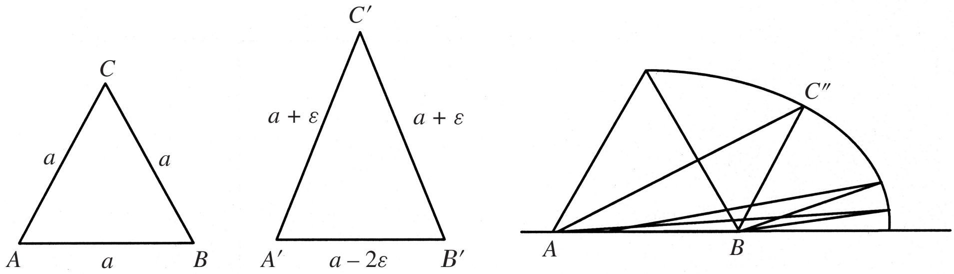

Figure 2.

There is also a constant-perimeter proof that involves only isosceles triangles. Starting with the equilateral triangle $\triangle ABC$ as before, we now let the apex (A' in Fig. 3) move down towards the midpoint of the base BC, keeping the perimeter fixed. As it does this, the areas of the isosceles triangles formed cover all values from S to 0 (by continuity). Now let the apex (A'' in Fig. 3) move up from A to a height of $3a/2$. Again, the areas of the isosceles triangles so formed cover all values from S to 0. Therefore, for every value R between S and 0, there are two isosceles triangles having perimeter P and area R, one steeper than $\triangle ABC$ and the other flatter.

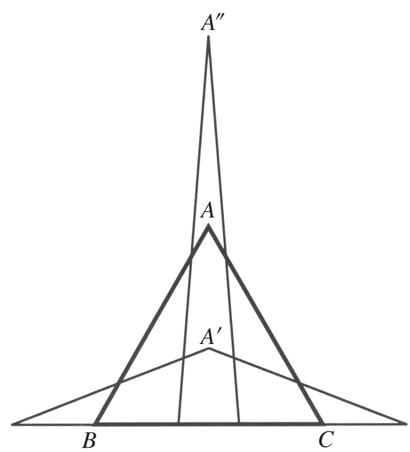

Figure 3.

Fixed area solutions. We now give similar constructions in which the area, rather than the perimeter, is kept fixed. This can be done by multiplying the base by a factor k and multiplying the height by $1/k$. Let $\triangle A'B'C'$ again denote the new triangle, and let P' denote its perimeter. As before, either the perimeters of the two triangles are equal or they are not. If they are equal, then we already have two non-congruent triangles with equal areas and equal perimeters. If the perimeters are not equal, then either $P' > P$ or $P > P'$. Without loss of generality, let us assume that $P' > P$ and see where this leads us. Again we change the original equilateral triangle, but this time by sliding the vertex A along the line through A parallel to the base BC. Any point on this line is the third vertex of a triangle whose base is a and whose area is S. The perimeters of these triangles are greater than P and get arbitrarily large. As before, it follows from the continuity principle that one of these triangles has area P';

it is clearly not congruent to $\triangle A'B'C'$, and so we have shown that there must be two different triangles with the same perimeters and the same areas.

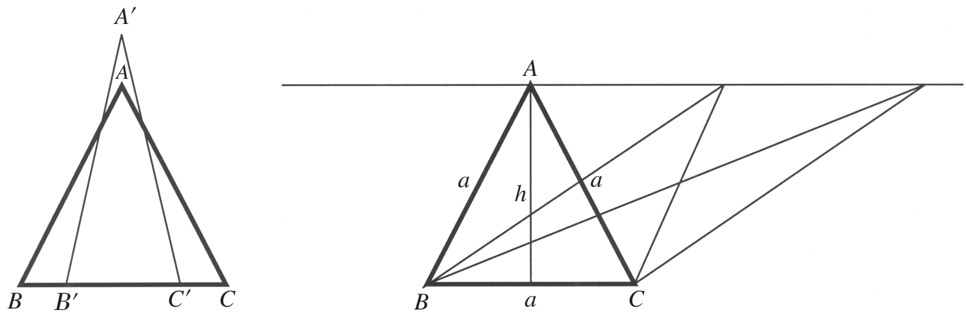

Figure 4.

I should like to note in closing that our isosceles triangle argument, in which the triangles all have the same perimeter and the area varies, can be modified to isosceles triangles for which the area is constant and the perimeter varies. The details are left to the reader.

Note. The author wishes to thank the editors for suggesting some of the ideas presented here.

References

1. R. Courant and H. Robbins, *What Is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford University Press, 1978.
2. H. J. Jacobs, *Geometry*, 2nd ed., Freeman, 2001.

Can You Paint a Can of Paint?

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The paradox of “Gabriel’s horn” is a favorite topic of many calculus teachers. (See, for example, [**1**, p. 402].) In this note, I offer two ways to resolve the paradox.

The “horn” is the surface S gotten by revolving the curve $y = 1/x$ for $x \geq 1$ about the x-axis. The surface has infinite area, but the volume of the 3-dimensional region R inside it is finite. Consider how surprising that is: a finite volume of paint is sufficient to fill R, and then every point of S will be in contact with paint—yet no quantity of paint, however large, will be enough to cover S with paint!

Certainly S has infinite area while R has finite volume—the calculations that lead to those results are not in doubt—so any resolution of the paradox requires us to question the interpretation of the calculations. I will argue that the paradox arises when we make erroneous assumptions about the relationships between area and paint, and between area and volume. Once those relationships are clarified, the apparent contradiction dissolves.

