The following "Three Angle Theorem" seems to be new.

Theorem 1. *Let $f(x)$ be a real function continuous on $[a, b]$ and differentiable on (a, b). Then there is a point c in (a, b) such that*

$$\tan\alpha(c) + \tan\beta(c) + \tan\gamma(c) = 0.$$

Proof. It is easily seen that

$$\tan\alpha(x) = f'(x);$$
$$\tan\beta(x) = \frac{f(x)}{x-a};$$
$$\tan\gamma(x) = \frac{f(x)}{x-b}.$$

Let $F(x) = f(x)(x-a)(x-b)$. Then it is easily checked that

1. $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
2. $F(a) = F(b) = 0$.

Therefore by Rolle's Theorem, there is a c in (a, b) such that $F'(c) = 0$.

Notice that $F'(x) = f'(x)(x-a)(x-b) + f(x)(x-b) + f(x)(x-a)$. By evaluating at $x = c$, we have

$$f'(c)(c-a)(c-b) + f(c)(c-b) + f(x)(c-a) = 0$$

or

$$f'(c) + \frac{f(c)}{c-a} + \frac{f(c)}{c-b} = 0.$$

Hence

$$\tan\alpha(c) + \tan\beta(c) + \tan\gamma(c) = 0. \qquad \blacksquare$$

Acknowledgment. The author thanks the referee sincerely for the suggestions improving this note.

A Tale of Two Tickets

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Lotto-type lottery games, in which players choose k of the numbers from 1 through n, are useful in the classroom as they provide "real-world" illustrations of many of the basic concepts from probability—elementary counting techniques, sample space, classical probability, odds, etc. Furthermore, many students feel a sense of satisfaction when their computations yield the same results as those stated on the playslips. In this capsule we use the Missouri Lotto and the stated odds on the playslips to provide a meaningful illustration of the rule for the probability of the union of two events. This example led to a very surprising result.

Before proceeding with our story, we mention a relevant point concerning the chances of winning a prize, as printed on the playslips from Missouri lotteries (and from most other states' as well). If p is the probability of winning a particular prize, and if m is obtained by rounding $1/p$ to the nearest integer, then the chance of winning is reported as $m : 1$, and is called the *odds of winning*.

Our saga begins with a certain professor who was discussing the Missouri Lotto with his elementary probability and statistics class. In this particular lotto game, a player chooses 6 of 44 numbers. The professor asked the students to compute the probability of a ticket matching 5 of 6 of the winning numbers. This was a straightforward problem, and most students had no trouble determining the correct answer,

$$\Pr(\text{match 5 of 6}) = \frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} = \frac{57}{1764763}.$$

The professor then asked the students to determine the rounded integer value m in order to see if their answer matched the odds of winning printed on the playslips he had provided them. By taking the reciprocal of $\frac{57}{1764763}$, the students found that

$$\Pr(\text{match 5 of 6}) \approx \frac{1}{30960.75},$$

and correctly claimed that the odds of matching 5 of 6 numbers was, to the nearest integer, 30961 : 1. They then turned over their playslips only to discover that the odds 30961 : 1 were nowhere to be found. (So much for the aforementioned sense of satisfaction!) A subsequent look at the playslips revealed that in the Missouri Lotto, a player must buy *two* tickets at \$.50 each, and the odds are given *per \$1 play*. Since two games were played, the students suggested that

$$\Pr(\text{match 5 of 6}) = 2 \cdot \frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} = \frac{114}{1764763} \approx \frac{2}{30960.75} \approx \frac{1}{15480.38}.$$

The playslip stated the odds were 15480 : 1, and the students were indeed satisfied. At this point, class was over and everyone went home happy. Everyone but the professor, that is, who having read ahead in the text was aware of the need to consider the probability that *both* tickets matched 5 of the 6 winning numbers . . . .

We will refer to the 15480 : 1 computed above as the *approximate odds* of matching 5 of 6 numbers. If A is the event "first ticket matches 5 of 6 numbers," and B is the event "second ticket matches 5 of 6 numbers," then the approximate odds are computed from the approximate probability, $P(A \cup B) \approx P(A) + P(B)$. The odds computed from the exact probability, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, will be called the *correct odds*. The question bothering the professor as he left class that day was this: Is $P(A \cap B)$ in fact so small that the approximate and correct odds of winning would round to the same integer? Consideration of this question led to a rather counterintuitive result, as we shall soon see.

Note that the (conditional) probability that both tickets match 5 of the 6 winning numbers will be different for different pairs of tickets, depending on how many numbers the two tickets have in common. However, in order to compute the correct odds, we need a meaningful *unconditional* probability that both tickets match 5 of the 6 winning numbers. We will therefore assume that the two tickets have their numbers chosen at random (using a computer generated "Quick Pick" for example). In this case, the events A and B are independent and we find that

$$\Pr(A \cup B) = 2 \cdot \frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} - \left(\frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} \right)^2 = \frac{201179733}{3114388446169} \approx \frac{1}{15480.\mathbf{62}}.$$

Thus while the approximate and exact probabilities are indeed very close, it is *not* the case that the corresponding odds of winning round to the same integer, and it is our belief that the odds should be stated as 15481 : 1 instead of 15480 : 1.* Of course the practical difference between these numbers is negligible—if you expect to wait 15480 weeks before winning such a prize, what's another week? More interesting than this particular example is the following theorem, which shows that for a lotto game of any size, the approximate odds and the correct odds will never differ by more than 1.

Theorem. Suppose we have a game in which a player choosing k of n numbers purchases two tickets, selecting the numbers at random. Let A be the event "first ticket matches j of n numbers," and B be the event "second ticket matches j of n numbers." Let the approximate probability $P(A \cup B) \approx P(A) + P(B)$ be expressed as $s : 1$, and the correct probability $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ be expressed as $t : 1$, where s and t are rounded to the nearest integer. Then $t - s = 0$ or $t - s = 1$.

Proof. Let $p = P(A) = P(B)$, and note that the events A and B are independent. Then

$$s = \text{Round}\left[\frac{1}{2p}\right] \quad \text{and} \quad t = \text{Round}\left[\frac{1}{2p - p^2}\right].$$

Since

$$\frac{1}{2p - p^2} - \frac{1}{2p} = \frac{1}{4 - 2p},$$

and since for $0 < p < 1$,

$$\frac{1}{4} < \frac{1}{4 - 2p} < \frac{1}{2},$$

we see that

$$\frac{1}{2p} \quad \text{and} \quad \frac{1}{2p - p^2}$$

differ by less than 1. It follows that $t - s = 0$ or $t - s = 1$. ■

Although we know of no lottery game that states the odds based on the purchase of more than two tickets, it is interesting to note that no matter how many tickets might hypothetically be required, the approximate odds and the correct odds will never differ by more than 1, a result we found startling. We state this formally, but we leave the proof as an exercise for the interested reader.

*It is interesting to note that a similar discrepancy occurs in Wisconsin's Very Own Megabucks game, but Wisconsin, which also follows the convention of rounding to the nearest integer, states the correct odds rather than the approximate.

Generalization. Suppose we have a game in which a player choosing k of n numbers purchases r tickets, selecting numbers at random. For $i = 1$ to r, let A_i be the event "ith ticket matches j out of n numbers." Let the approximate probability $P(A_1 \cup A_2 \cup \cdots \cup A_r) \approx \sum_{i=1}^{r} P(A_i)$ be expressed as $s : 1$, and let the exact probability for $P(A_1 \cup A_2 \cup \cdots \cup A_r)$ using inclusion/exclusion be expressed as $t : 1$, where s and t are rounded to the nearest integer. Then $t - s = 0$ or $t - s = 1$.

The Chain Rule for Matrix Exponential Functions

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This short note serves as an extension of Liu's note [**4**]. The problem is to determine the extent to which the chain rule for scalar exponential functions (i.e., $(\exp(f(t)))' = \exp(f(t))f'(t)$) extends to the context of matrix exponential functions.

If A is an $n \times n$ matrix, it is well known ([**2**], [**3**]) that the series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

(I denoting the $n \times n$ identity matrix) converges to an $n \times n$ matrix denoted by $\exp(A)$. One can then prove (see [**3**]) that

$$\exp(tA)' = A\exp(tA) = \exp(tA)A. \tag{1}$$

(All derivatives will be with respect to a real parameter t.) The question is whether the chain rule (1) extends to more general matrix exponential functions than just $\exp(tA)$. That is, if $B = B(t)$ is an $n \times n$ matrix of differentiable functions, is it true that

$$\exp(B)' = B'\exp(B) = \exp(B)B'?$$

Equation (1) says that the answer is 'yes' if B has the form $B = tA$, where A is a matrix of constants.

In general the answer is 'no.' Liu provided a counter-example in [**4**]. A more conceptual explanation is that matrix exponential manipulations do not work as in the scalar case unless the matrices involved commute. Such is the situation with the chain rule problem here.

Exercise 1. For any fixed value of θ, set

$$A = \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & -\cos\theta \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}.$$

Show that $A^3 = -A$, and that, for any value of t, $\exp(tA) = I + (\sin t)A + (1 - \cos t)A^2$.

Exercise 2. If A_1 and A_2 are $n \times n$ matrices, then $(A_1 + A_2)^2 = A_1^2 + 2A_1A_2 + A_2^2$ if and only if A_1 and A_2 commute.

