CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

An Apothem Apparently Appears

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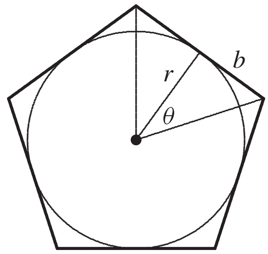
Since the number of distinctly different optimization problems in differential calculus is quite limited, we were rather surprised to discover an apparently unnoticed property in the familiar "cut a wire" problem. It is a beautiful and elegant result and is easy to spot once you are told where to look. Furthermore, the result extends to the case in which there are more than two figures and also to several collections of figures in three dimensions. Rather than give away the solution in the introduction (assuming the title has not already done so), we hope the reader will trust us enough to continue.

As a start, we recommend that the reader solve all three of the following wire problems; at least one of these problems appears in most current calculus textbooks.

> A wire of length L is cut into two pieces. One piece is bent into a circle and the other piece bent into a square. How should the wire be cut in order to minimize the sum of the enclosed areas? Repeat the problem if the pieces are bent into a circle and an equilateral triangle or a square and an equilateral triangle.

When finished, look at your solutions and compare the dimensions of the optimal figures. If you solved the problem in the standard way, there seems to be little worth noting. With a closer look, you might notice that the circle is inscribed in the square, the circle is inscribed in the equilateral triangle, and the square and equilateral triangle have the same inscribed circle. In other words, in each case the optimal figures have the same apothem (the minimum distance from the center of the figure to an edge). We will show that this property of the solution extends to any pair of regular polygons, then consider some extensions of this result.

To consider the wire problem for arbitrary regular polygons, we need convenient formulas for their area and perimeter. For each positive integer n greater than 2, let $\alpha_n = n\tan(\pi/n)$. It is elementary to verify that the perimeter P and area A of a regular n-gon with apothem r are given by $P = 2\alpha_n r$ and $A = \alpha_n r^2$. The figure shows a regular pentagon, but gives the general calculations.



For an arbitrary n-gon,

$\theta = \pi/n;$
$b = r\tan\theta;$
$P = n \cdot 2b = 2\alpha_n r;$
$A = n \cdot br = \alpha_n r^2.$

Since a circle can be thought of as a regular ∞-gon (the sequence $\{\alpha_n\}$ converges to π), these formulas include the special case of a circle if we set $\alpha_\infty = \pi$. Note that $P = dA/dr$; perimeter is the rate of change of area with respect to the apothem. Students frequently observe this property of circles, but then realize it does not carry over to the usual formulas for a square. Some comments concerning this fact will be made at the end of the paper.

We can now pose the wire problem in a more general setting. Let p and q represent either ∞ or an integer greater than 2 ($p = q$ is allowed) and let L be a fixed positive constant. Suppose that the sum of the perimeters of a regular p-gon and a regular q-gon is L. We claim that the minimum area enclosed by the two polygons occurs when they have the same apothem. Let x be the apothem of the regular p-gon and let y be the apothem of the regular q-gon. The relevant optimization problem is to minimize $Q = \alpha_p x^2 + \alpha_q y^2$ subject to $x \geq 0$, $y \geq 0$, and $2\alpha_p x + 2\alpha_q y = L$. Although this problem can be solved easily with or without calculus, we will include the details of a calculus solution that is somewhat different from the usual methods. Differentiate the constraint equation implicitly with respect to x to obtain $dy/dx = -\alpha_p/\alpha_q$. It follows that

$$\frac{dQ}{dx} = 2\alpha_p x + 2\alpha_q y\,\frac{dy}{dx} = 2\alpha_p x - 2\alpha_q y\,\frac{\alpha_p}{\alpha_q} = 2\alpha_p(x - y).$$

Thus, the quantity Q has a possible extreme value when $x = y$, that is, when the two polygons have the same apothem. Referring to the following data,

i. if $x = 0$ and $y = \dfrac{L/2}{\alpha_q}$, then $Q = \dfrac{1}{\alpha_q}\left(\dfrac{L}{2}\right)^2$;

ii. if $x = \dfrac{L/2}{\alpha_p + \alpha_q}$ and $y = \dfrac{L/2}{\alpha_p + \alpha_q}$, then $Q = \dfrac{1}{\alpha_p + \alpha_q}\left(\dfrac{L}{2}\right)^2 = Q_{\min}$;

iii. if $x = \dfrac{L/2}{\alpha_p}$ and $y = 0$, then $Q = \dfrac{1}{\alpha_p}\left(\dfrac{L}{2}\right)^2$;

it is clear that the minimum value of Q occurs when $x = y$. Note the symmetry in the values of Q at the endpoints and the critical point. When Q is a minimum, the perimeter and area of the two figures are

$$p\text{-gon,}\qquad P = 2\alpha_p x = \frac{\alpha_p}{\alpha_p + \alpha_q}\cdot L;\qquad A = \alpha_p x^2 = \frac{\alpha_p}{\alpha_p + \alpha_q}\cdot Q_{\min};$$

$$q\text{-gon,}\qquad P = 2\alpha_q y = \frac{\alpha_q}{\alpha_p + \alpha_q}\cdot L;\qquad A = \alpha_q y^2 = \frac{\alpha_q}{\alpha_p + \alpha_q}\cdot Q_{\min}.$$

Hence, the perimeters and areas of the optimal figures are distributed in the same proportions.

We next generalize this result to the case in which there are more than two figures, that is, the wire is cut into several pieces and each one bent into a regular polygon of some type. Let $\mathcal{R}$ denote the collection of all circles and all regular polygons. Let F be a geometric figure in the collection $\mathcal{R}$ and denote its apothem by r. Then (as shown above) the perimeter P_F and area A_F of F are given by $P_F = 2c\,r$ and $A_F = c\,r^2$, where c is a fixed proportionality constant that depends on F. Let n be a positive integer greater than 1 and suppose that the sum of the perimeters of n geometric figures chosen from the collection $\mathcal{R}$ is a fixed constant L. We claim that the minimum area enclosed by these figures occurs when they all have the same apothem and that the perimeters and areas of the optimal figures are distributed in the same proportions. Let x_k be the apothem of the kth figure and let c_k be the corresponding constant of proportionality. The relevant optimization problem is to minimize $Q = \sum_{k=1}^{n} c_k\, x_k^2$ subject to $x_k \geq 0$ for all k and $\sum_{k=1}^{n} 2c_k x_k = L$. Using the technique of Lagrange multipliers, it is elementary to verify that the minimum occurs when all of the figures have the same apothem. Let r be this common value and let $\sigma = \sum_{k=1}^{n} c_k$. Then $r = L/(2\sigma)$ and $Q_{\min} = \sigma r^2 = (L/2)^2/\sigma$. The perimeter P_k and area A_k of the kth figure are then

$$P_k = 2c_k r = \frac{c_k}{\sigma}\, L, \qquad A_k = c_k r^2 = \frac{c_k}{\sigma}\, Q_{\min}.$$

This establishes the claim. So if you are faced with the following nightmarish calculus problem:

> A wire of length 70 meters is to be cut into 35 pieces and the pieces are bent to form 2 circles, 3 equilateral triangles, 4 squares, 5 regular pentagons, 6 regular hexagons, 7 regular heptagons, and 8 regular octagons. How should the wire be cut in order to minimize the sum of the enclosed areas?

you can give a quick and simple answer; find the common value of the apothem, then circumscribe all of the polygons around the circle with that radius.

The wire problem can easily be extended to three dimensions; fix the surface area and minimize the sum of the enclosed volumes. The main dilemma is what figures to consider so that the solution has nice features. The most natural extension is to regular solids. Let $\mathcal{S}$ denote the collection of all spheres and all regular solids. There are only five of these regular solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. One of many sources for information on these is [**5**]. (The advantage of a web site for this topic is the colorful pictures.) Let F be a geometric figure in the collection $\mathcal{S}$ and denote the radius of its inscribed sphere by r. Then the surface area S_F and volume V_F of F are given by $S_F = 3c\,r^2$ and $V_F = c\,r^3$, where c is a fixed proportionality constant that depends on F. (Note once again the interesting relationship $S = dV/dr$.) These constants are given in the following table.

figure	proportionality constant	numerical approximation
tetrahedron	$8\sqrt{3}$	13.8564
cube	8	8.0000
octahedron	$4\sqrt{3}$	6.9282
dodecahedron	$10\sqrt{130 - 58\sqrt{5}}$	5.5503
icosahedron	$10\sqrt{3}\left(7 - 3\sqrt{5}\right)$	5.0541
sphere	$4\pi/3$	4.1888

